VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2011

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Letter

Figures

Words

MATHEMATICAL METHODS (CAS)

Written examination 2

Wednesday 9 November 2011

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer-based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 25 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The midpoint of the line segment joining (0, –5) to (d, 0) is

A. $\left(\frac{d}{2}, -\frac{5}{2}\right)$

B. $(0, 0)$

C. $\left(\frac{d-5}{2}, 0\right)$

D. $\left(0, \frac{5-d}{2}\right)$

E. $\left(\frac{5+d}{2}, 0\right)$

Question 2

The gradient of a line **perpendicular** to the line which passes through (–2, 0) and (0, –4) is

A. $\frac{1}{2}$

B. –2

C. $-\frac{1}{2}$

D. 4

E. 2

Question 3

If $x + a$ is a factor of $4x^3 - 13x^2 - ax$, where $a \in R\backslash\{0\}$, then the value of a is

A. –4
B. –3
C. –1
D. 1
E. 2

Question 4

The derivative of $\log_e(2f(x))$ with respect to x is

A. $\dfrac{f'(x)}{f(x)}$

B. $2\dfrac{f'(x)}{f(x)}$

C. $\dfrac{f'(x)}{2f(x)}$

D. $\log_e(2f'(x))$

E. $2\log_e(2f'(x))$

Question 5

The inverse function of $g: [2, \infty) \to R, g(x) = \sqrt{2x-4}$ is

A. $g^{-1}: [2, \infty) \to R, g^{-1}(x) = \dfrac{x^2+4}{2}$

B. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = (2x-4)^2$

C. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \sqrt{\dfrac{x}{2}+4}$

D. $g^{-1}: [0, \infty) \to R, g^{-1}(x) = \dfrac{x^2+4}{2}$

E. $g^{-1}: R \to R, g^{-1}(x) = \dfrac{x^2+4}{2}$

Question 6

For the continuous random variable X with probability density function

$$f(x) = \begin{cases} \log_e(x) & 1 \le x \le e \\ 0 & elsewhere \end{cases}$$

the expected value of X, $E(X)$, is closest to

A. 0.358

B. 0.5

C. 1

D. 1.859

E. 2.097

Question 7

By considering the point (8, 2) on the graph of $f(x) = \sqrt[3]{x}$, and using the linear approximation method, $f(8 + h) \approx f(8) + hf'(8)$, an estimate for $\sqrt[3]{8.5}$ is

A. 2.5000

B. 2.08333

C. 2.04167

D. 2.04083

E. 2.00347

Question 8

Consider the function $f: R \to R, f(x) = x(x - 4)$ and the function

$$g: \left[\frac{3}{2}, 5\right) \to R, g(x) = x + 3.$$

If the function $h = f + g$, then the domain of the inverse function of h is

A. $[0, 13)$

B. $\left[-\frac{3}{4}, 10\right]$

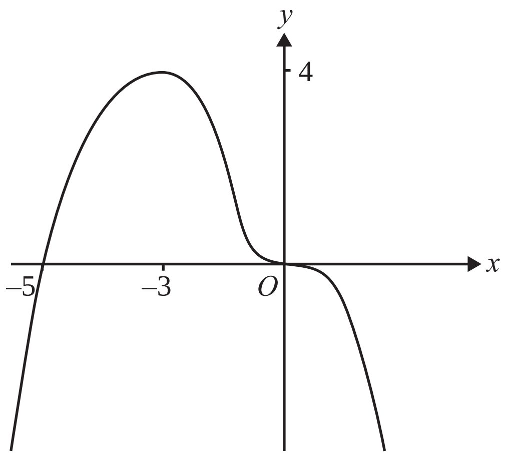
C. $\left(-\frac{3}{4}, \frac{15}{4}\right]$

D. $\left[\frac{3}{4}, 13\right)$

E. $\left[\frac{3}{2}, 13\right)$

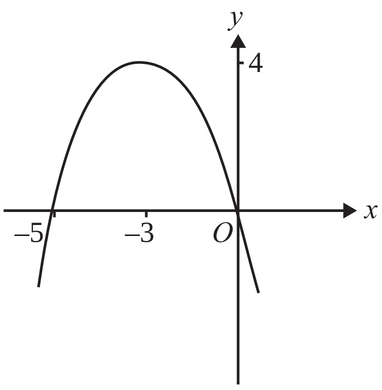
Question 9

The graph of the function $y = f(x)$ is shown below.

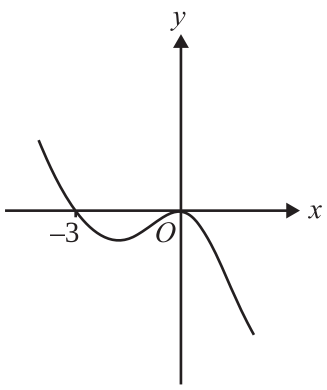


Which of the following could be the graph of the derivative function $y = f'(x)$?

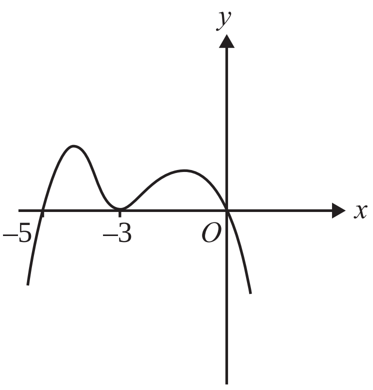
A.



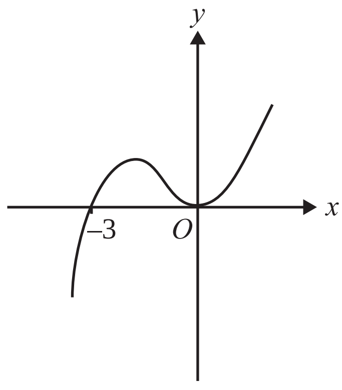
B.



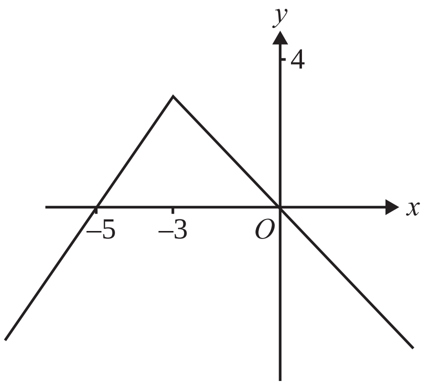
C.



D.



E.



Question 10

Maria is a very enthusiastic young tennis player who plays a match every day. If she wins on one particular day, the probability that she wins the next day is 0.8. If she loses one day, the probability that she loses the next day is 0.6.

The long-term probability that Maria wins a match is

A. $\frac{4}{5}$

B. $\frac{2}{5}$

C. $\frac{2}{3}$

D. $\frac{1}{3}$

E. $\frac{3}{4}$

Question 11

The average value of the function with rule $f(x) = \log_e(x + 2)$ over the interval [0, 3] is

A. $\log_e(2)$

B. $\frac{1}{3}\log_e(6)$

C. $\log_e\left(\frac{3125}{4}\right) - 3$

D. $\frac{1}{3}\log_e\left(\frac{3125}{4}\right) - 3$

E. $\frac{5\log_e(5) - 2\log_e(2) - 3}{3}$

Question 12

The continuous random variable X has a normal distribution with mean 30 and standard deviation 5. For a given number a, $\Pr(X > a) = 0.20$.

Correct to two decimal places, a is equal to

A. 23.59

B. 24.00

C. 25.79

D. 34.21

E. 36.41

Question 13

In an orchard of 2000 apple trees it is found that 1735 trees have a height greater than 2.8 metres. The heights are distributed normally with a mean μ and standard deviation 0.2 metres.

The value of μ is closest to

A. 3.023

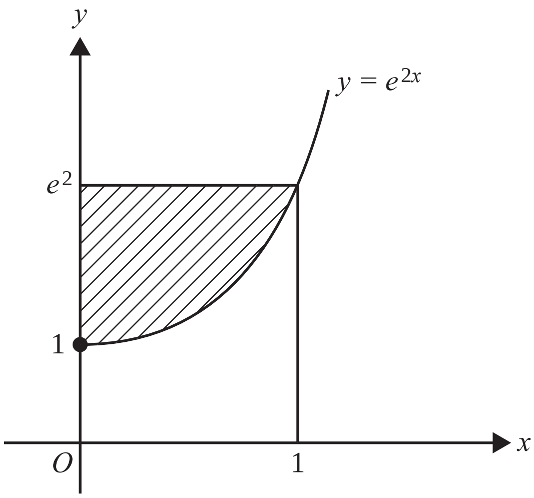
B. 2.577

C. 2.230

D. 1.115

E. 0.223

Question 14



To find the area of the shaded region in the diagram shown, four different students proposed the following calculations.

i. $\int_0^1 e^{2x}\,dx$

ii. $e^2 - \int_0^1 e^{2x}\,dx$

iii. $\int_1^{e^2} e^{2y}\,dy$

iv. $\int_1^{e^2} \frac{\log_e(x)}{2}\,dx$

Which of the following is correct?

A. **ii.** only

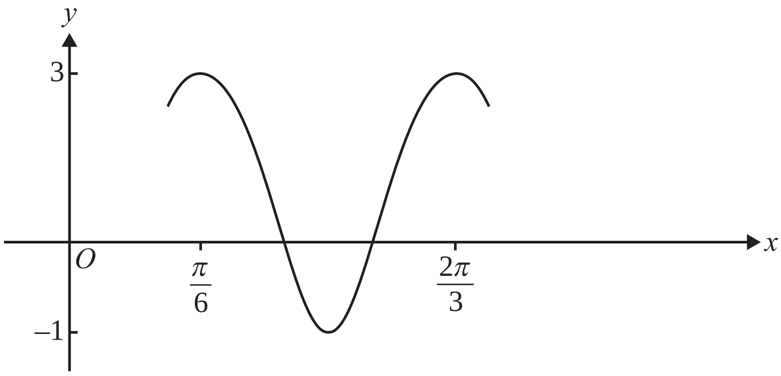
B. **ii.** and **iii.** only

C. **i.**, **ii.**, **iii.** and **iv.**

D. **ii.** and **iv.** only

E. **i.** and **iv.** only

Question 15



The graph shown could have equation

A. $y = 2\cos\left(x + \frac{\pi}{6}\right) + 1$

B. $y = 2\cos 4\left(x - \frac{\pi}{6}\right) + 1$

C. $y = 4\sin 2\left(x - \frac{\pi}{12}\right) - 1$

D. $y = 3\cos\left(2x + \frac{\pi}{6}\right) - 1$

E. $y = 2\sin\left(4x + \frac{2\pi}{3}\right) - 1$

Question 16

Which of the following is **not** true about the function

$$f: R \to R, f(x) = |x^2 - 4| - 2?$$

A. The graph of f is continuous everywhere.
B. $f(-2) = -2$.
C. $f(x) \geq -2$ for all values of x.
D. $f'(x) = 2x$ for all $x > 0$.
E. $f'(x) = 2x$ for all $x < -2$.

Question 17

The normal to the curve with equation $y = x^{\frac{3}{2}} + x$ at the point (4, 12) is parallel to the straight line with equation

A. $4x = y$

B. $4y + x = 7$

C. $y = \frac{x}{4} + 1$

D. $x - 4y = -5$

E. $4y + 4x = 20$

Question 18

The equation $x^3 - 9x^2 + 15x + w = 0$ has only one solution for x when

A. $-7 < w < 25$

B. $w \leq -7$

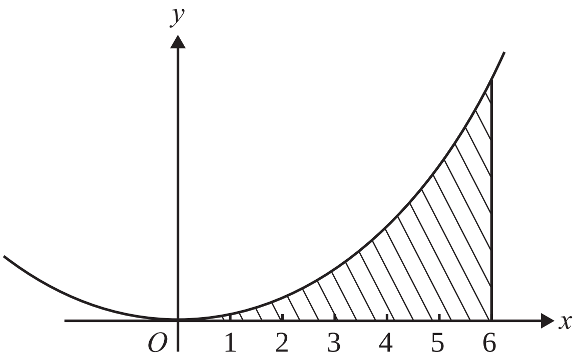
C. $w \geq 25$

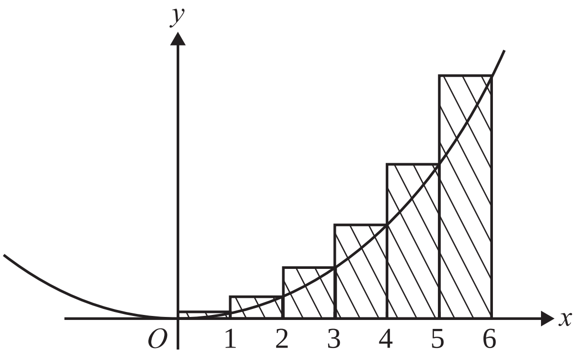
D. $w < -7$ or $w > 25$

E. $w > 1$

Question 19

A part of the graph of $f: R \rightarrow R, f(x) = x^2$ is shown below. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.





Zoe's approximation is p% more than the exact value of the area.

The value of p is closest to

A. 10

B. 15

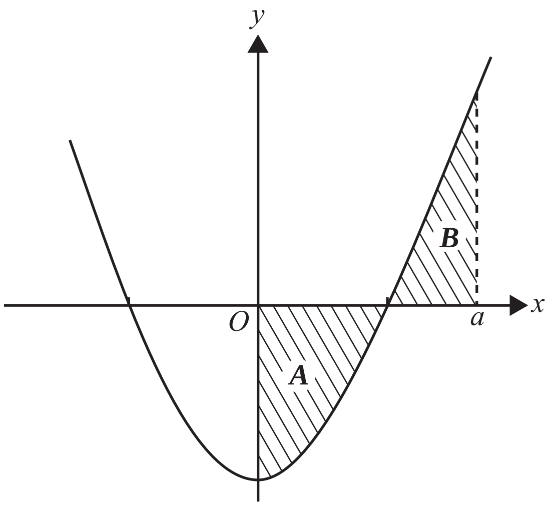
C. 20

D. 25

E. 30

Question 20

A part of the graph of $g: R \rightarrow R, g(x) = x^2 - 4$ is shown below.



The area of the region marked ***A*** is the same as the area of the region marked ***B***.

The exact value of a is

A. 0

B. 6

C. $\sqrt{6}$

D. 12

E. $2\sqrt{3}$

Question 21

For two events, P and Q, $\Pr(P \cap Q) = \Pr(P' \cap Q)$.

P and Q will be independent events exactly when

A. $\Pr(P') = \Pr(Q)$

B. $\Pr(P \cap Q') = \Pr(P' \cap Q)$

C. $\Pr(P \cap Q) = \Pr(P) + \Pr(Q)$

D. $\Pr(P \cap Q') = \Pr(P \cap Q)$

E. $\Pr(P) = \frac{1}{2}$

Question 22

The expression

$$\log_c(a) + \log_a(b) + \log_b(c)$$

is equal to

A. $\dfrac{1}{\log_c(a)} + \dfrac{1}{\log_a(b)} + \dfrac{1}{\log_b(c)}$

B. $\dfrac{1}{\log_a(c)} + \dfrac{1}{\log_b(a)} + \dfrac{1}{\log_c(b)}$

C. $-\dfrac{1}{\log_a(b)} - \dfrac{1}{\log_b(c)} - \dfrac{1}{\log_c(a)}$

D. $\dfrac{1}{\log_a(a)} + \dfrac{1}{\log_b(b)} + \dfrac{1}{\log_c(c)}$

E. $\dfrac{1}{\log_c(ab)} + \dfrac{1}{\log_b(ac)} + \dfrac{1}{\log_a(cb)}$

END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Two ships, the Elsa and the Violet, have collided. Fuel immediately starts leaking from the Elsa into the sea.

The captain of the Elsa estimates that at the time of the collision his ship has 6075 litres of fuel on board and he also forecasts that it will leak into the sea at a rate of $\frac{t^2}{5}$ litres per minute, where t is the number of minutes that have elapsed since the collision.

a. At this rate how long, in minutes, will it take for all the fuel from the Elsa to leak into the sea?

3 marks

In fact, the captain's assessment of the situation is wrong. The amount of fuel on board at the time of the collision **is not** 6075 litres. The fuel actually leaks into the sea forming a circular oil slick. The area of this circle is increasing at the constant rate of 20 square metres per minute.

At time t minutes after the collision the radius of the circle is r metres.

b. Show that when the radius of the circle is 3 metres, the radius is increasing at $\frac{10}{3\pi}$ metres per minute.

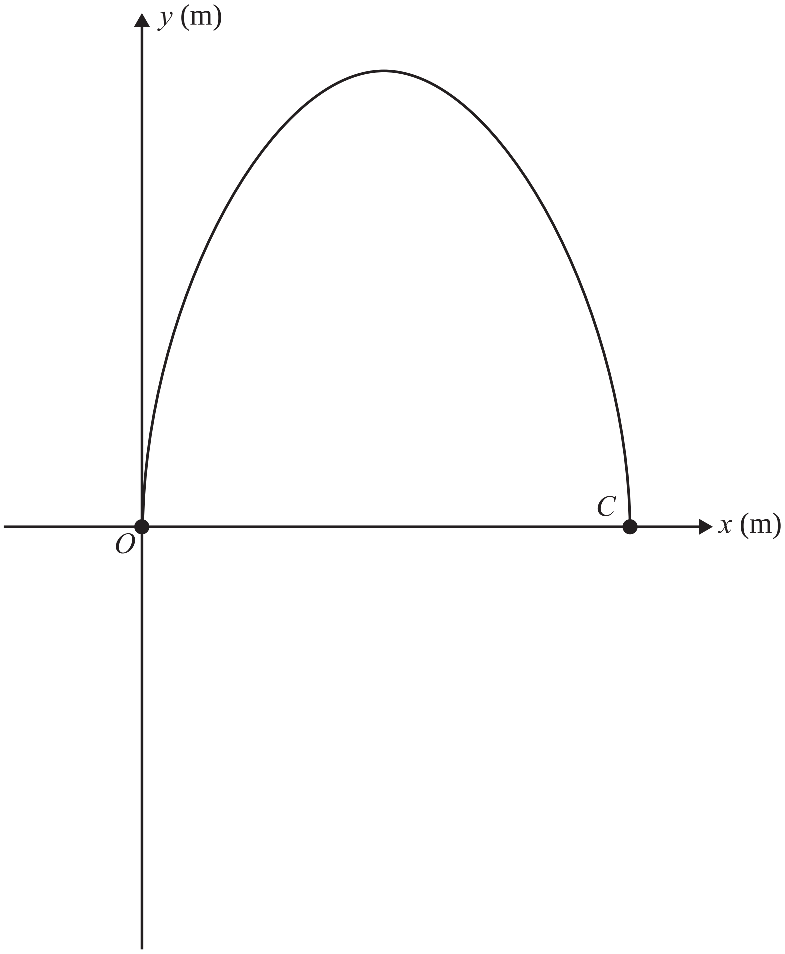
3 marks

In the following, all measurements of distance are in metres.

Sometime later, the fuel stops leaking. Authorities will use two floating barriers to surround the oil slick. One of the barriers, barrier 1, is in the shape of the graph of the function

$$g: [0, 80] \rightarrow R,\ g(x) = 75\sin\left(\frac{\pi x}{80}\right).$$

This graph is shown below.



The second barrier enclosing the oil slick, barrier 2, is in the shape of the graph of the function h given by

$$h: [0, 80] \rightarrow R,\ h(x) = -60\left|\sin\left(\frac{\pi x}{40}\right)\right|.$$

c. On the axes above, sketch the graph of $y = h(x)$. Label all axes intercepts and turning points with their coordinates.

2 marks

d. Some time later, the oil slick completely fills the area enclosed by the two barriers.

i. Find $\int_0^{40} \sin\left(\frac{\pi x}{40}\right) dx$

ii. Find, to the nearest square metre, the total area of the oil slick at that time.

1 + 2 = 3 marks

Total 11 marks

Question 2

In a chocolate factory the material for making each chocolate is sent to one of two machines, machine *A* or machine *B*.

The time, *X* seconds, taken to produce a chocolate by machine *A,* is normally distributed with mean 3 and standard deviation 0.8.

The time, *Y* seconds, taken to produce a chocolate by machine *B*, has the following probability density function.

$$f(y) = \begin{cases} 0 & y < 0 \\ \dfrac{y}{16} & 0 \le y \le 4 \\ 0.25e^{-0.5(y-4)} & y > 4 \end{cases}$$

a. Find correct to four decimal places

i. $\Pr(3 \le X \le 5)$

ii. $\Pr(3 \le Y \le 5)$

1 + 3 = 4 marks

b. Find the mean of *Y*, correct to three decimal places.

3 marks

c. **i.** Find the median of Y.

ii. Find the value of a, correct to two decimal places, such that $\Pr(Y \le a) = 0.7$.

1 + 2 = 3 marks

d. It can be shown that $\Pr(Y \le 3) = \frac{9}{32}$. A random sample of 10 chocolates **produced by machine *B*** is chosen.

Find the probability, correct to four decimal places, that exactly 4 of these 10 chocolates took 3 or less seconds to produce.

2 marks

All of the chocolates produced by machine A and machine B are stored in a large bin. There is an equal number of chocolates from each machine in the bin.

It is found that if a chocolate, produced by either machine, takes longer than 3 seconds to produce then it can easily be identified by its darker colour.

e. A chocolate is selected at random from the bin. It is found to have taken longer than 3 seconds to produce. Find, correct to four decimal places, the probability that it was produced by machine A.

3 marks

Total 15 marks

CONTINUES OVER PAGE

Question 3

a. Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9$.

i. Find $f'(x)$

ii. Explain why $f'(x) \geq 5$ for all x.

1 + 1 = 2 marks

b. The cubic function p is defined by $p: R \to R, p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers.

i. If p has m stationary points, what possible values can m have?

ii. If p has an inverse function, what possible values can m have?

1 + 1 = 2 marks

c. The cubic function q is defined by $q: R \to R, q(x) = 3 - 2x^3$.

i. Write down an expression for $q^{-1}(x)$.

ii. Determine the coordinates of the point(s) of intersection of the graphs of $y = q(x)$ and $y = q^{-1}(x)$.

2 + 2 = 4 marks

d. The cubic function g is defined by $g: R \to R, g(x) = x^3 + 2x^2 + cx + k$, where c and k are real numbers.

i. If g has exactly one stationary point, find the value of c.

ii. If this stationary point occurs at a point of intersection of $y = g(x)$ and $y = g^{-1}(x)$, find the value of k.

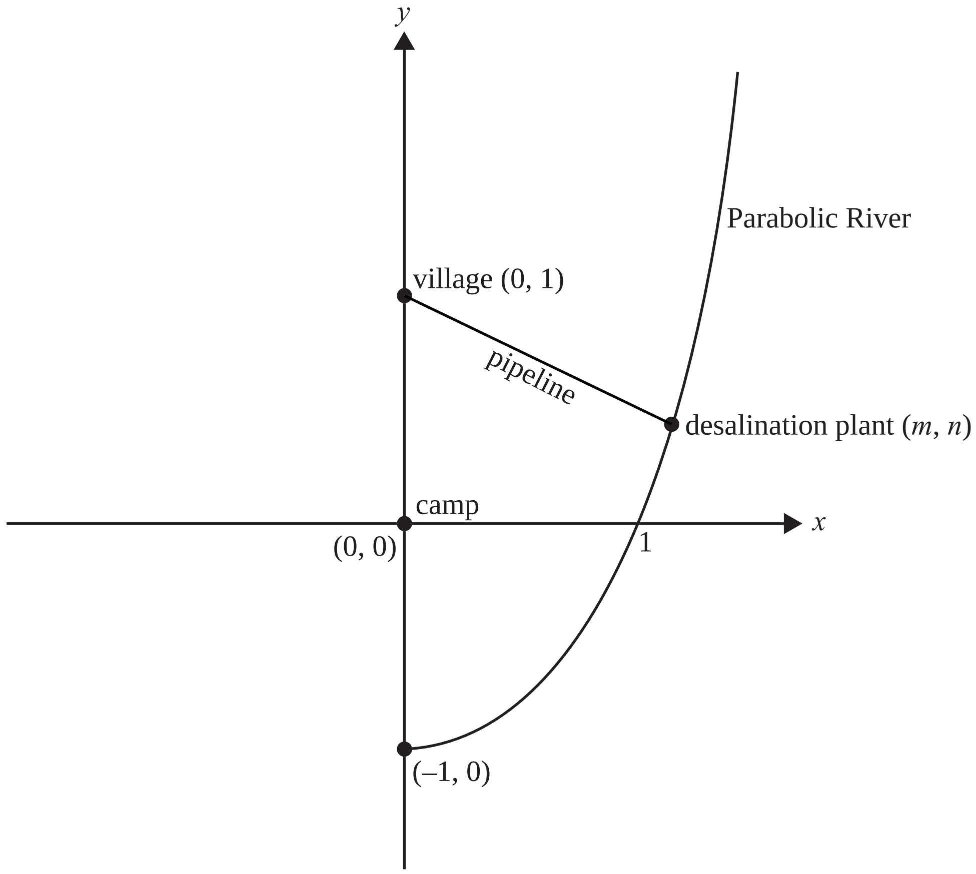
3 + 3 = 6 marks

Total 14 marks

Question 4

Deep in the South American jungle, Tasmania Jones has been working to help the Quetzacotl tribe to get drinking water from the very salty water of the Parabolic River. The river follows the curve with equation $y = x^2 - 1, x \geq 0$ as shown below. All lengths are measured in kilometres.

Tasmania has his camp site at (0, 0) and the Quetzacotl tribe's village is at (0, 1). Tasmania builds a desalination plant, which is connected to the village by a straight pipeline.



a. If the desalination plant is at the point (m, n) show that the length, L kilometres, of the straight pipeline that carries the water from the desalination plant to the village is given by

$$L = \sqrt{m^4 - 3m^2 + 4}.$$

3 marks

b. If the desalination plant is built at the point on the river that is closest to the village

i. find $\frac{dL}{dm}$ and hence find the coordinates of the desalination plant

ii. find the length, in kilometres, of the pipeline from the desalination plant to the village.

3 + 2 = 5 marks

The desalination plant is actually built at $\left(\frac{\sqrt{7}}{2}, \frac{3}{4}\right)$.

If the desalination plant stops working, Tasmania needs to get to the plant in the minimum time.

Tasmania runs in a straight line from his camp to a point (x, y) on the river bank where $x \leq \frac{\sqrt{7}}{2}$. He then swims up the river to the desalination plant.

Tasmania runs from his camp to the river at 2 km per hour. The time that he takes to swim to the desalination plant is proportional to the difference between the y-coordinates of the desalination plant and the point where he enters the river.

c. Show that the total time taken to get to the desalination plant is given by

$$T = \frac{1}{2}\sqrt{x^4 - x^2 + 1} + \frac{1}{4}k\left(7 - 4x^2\right)$$ hours where k is a positive constant of proportionality.

3 marks

The value of k varies from day to day depending on the weather conditions.

d. If $k = \frac{1}{2\sqrt{13}}$

i. find $\frac{dT}{dx}$

ii. hence find the coordinates of the point where Tasmania should reach the river if he is to get to the desalination plant in the minimum time.

1 + 2 = 3 marks

e. On one particular day, the value of k is such that Tasmania should run directly from his camp to the point $(1, 0)$ on the river to get to the desalination plant in the minimum time. Find the value of k on that particular day.

2 marks

f. Find the values of k for which Tasmania should run directly from his camp towards the desalination plant to reach it in the minimum time.

2 marks

Total 18 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.
This formula sheet is provided for your reference.

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Mathematical Methods (CAS)
Formulas

Mensuration

area of a trapezium: $\frac{1}{2}(a+b)h$

curved surface area of a cylinder: $2\pi rh$

volume of a cylinder: $\pi r^2 h$

volume of a cone: $\frac{1}{3}\pi r^2 h$

volume of a pyramid: $\frac{1}{3}Ah$

volume of a sphere: $\frac{4}{3}\pi r^3$

area of a triangle: $\frac{1}{2}bc\sin A$

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(e^{ax}) = ae^{ax}$

$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$

$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$

$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$

$\frac{d}{dx}(\tan(ax)) = \frac{a}{\cos^2(ax)} = a\sec^2(ax)$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\int \frac{1}{x} dx = \log_e|x| + c$

$\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + c$

$\int \cos(ax)dx = \frac{1}{a}\sin(ax) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

transition matrices: $S_n = T^n \times S_0$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2 p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET