VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2009

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

Letter

Figures

Words

MATHEMATICAL METHODS

Written examination 2

Monday 9 November 2009

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	4	4	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved **graphics** calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 22 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.

Choose the response that is **correct** for the question.

A correct answer scores 1, an incorrect answer scores 0.

Marks will **not** be deducted for incorrect answers.

No marks will be given if more than one answer is completed for any question.

Question 1

The point Q (2, 3) lies on the graph of a function f. The graph of f is translated 3 units up (parallel to the y-axis) and then reflected in the x-axis.

The coordinates of the final image of Q are

A. (2, 6)

B. (5, –3)

C. (–2, 6)

D. (2, –6)

E. (–5, 3)

Question 2

At the point (1, 1) on the graph of the function with rule $y = (x - 1)^3 + 1$

A. there is a local maximum.

B. there is a local minimum.

C. there is a stationary point of inflection.

D. the gradient is not defined.

E. there is a point of discontinuity.

Question 3

The maximal domain D of the function $f : D \to R$ with rule $f(x) = \log_e(2x + 1)$ is

A. $R \setminus \left\{-\frac{1}{2}\right\}$

B. $\left(-\frac{1}{2}, \infty\right)$

C. R

D. $(0, \infty)$

E. $\left(-\infty, -\frac{1}{2}\right)$

Question 4

If $f\colon R \to R$, $f(x) = \sin(2x) + 1$ and $g\colon R^+ \to R$, $g(x) = \log_e(x)$ then $f(g(x))$ is equal to

A. $\log_e(\sin(2x) + 1)$

B. $\sin(2\log_e(x) + 1)$

C. $(\sin(2x) + 1)\log_e(x)$

D. $\sin(\log_e(x^2)) + 1$

E. $\sin((\log_e(x))^2) + 1$

Question 5

The range of the function $f\colon [-2, 3) \to R$, $f(x) = 3x^2 - 12$ is

A. R

B. $[-12, \infty)$

C. $[0, 15)$

D. $[-12, 15)$

E. $(-\infty, 15)$

Question 6

The continuous random variable X has a normal distribution with mean 14 and standard deviation 2.

If the random variable Z has the standard normal distribution, then the probability that X is greater than 17 is equal to

A. $\Pr(Z > 3)$

B. $\Pr(Z < 2)$

C. $\Pr(Z < 1.5)$

D. $\Pr(Z < -1.5)$

E. $\Pr(Z > 2)$

Question 7

For $y = e^{2x}\cos(3x)$ the rate of change of y with respect to x when $x = 0$ is

A. 0

B. 2

C. 3

D. –6

E. –1

Question 8

For the function $f\colon R \to R$, $f(x) = (x + 5)^2(x - 1)$, the subset of R for which the gradient of f is negative is

A. $(-\infty, 1)$

B. $(-5, 1)$

C. $(-5, -1)$

D. $(-\infty, -5)$

E. $(-5, 0)$

Question 9

The tangent at the point (1, 5) on the graph of the curve $y = f(x)$ has equation $y = 3 + 2x$.

The tangent at the point (3, 8) on the curve $y = f(x - 2) + 3$ has equation

A. $y = 2x - 4$

B. $y = x + 5$

C. $y = -2x + 14$

D. $y = 2x + 4$

E. $y = 2x + 2$

Question 10

The discrete random variable X has a probability distribution as shown.

x	0	1	2	3
$\Pr(X = x)$	0.4	0.2	0.3	0.1

The median of X is

A. 0

B. 1

C. 1.1

D. 1.2

E. 2

Question 11

The continuous random variable X has a probability density function given by

$$f(x) = \begin{cases} \pi \sin(2\pi x) & \text{if } 0 \le x \le \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

The value of a such that $\Pr(X > a) = 0.2$ is closest to

A. 0.26

B. 0.30

C. 0.32

D. 0.35

E. 0.40

Question 12

The average rate of change of the function with rule $f(x) = x^3 - e^x$ between $x = 0$ and $x = 1$ is

A. 0

B. 2

C. $1 - e$

D. $2 - e$

E. $3 - e$

Question 13

A fair coin is tossed twelve times.

The probability (correct to four decimal places) that at most 4 heads are obtained is

A. 0.0730

B. 0.1209

C. 0.1938

D. 0.8062

E. 0.9270

Question 14

Which one of the following is **not** true for the function with rule $f(x) = \left|x^{\frac{3}{5}}\right| + 2$?

A. $f(32) = 10$.

B. The gradient of the function at the point (0, 2) is not defined.

C. $f(x) \geq 2$ for all real values of x.

D. There is a stationary point where $x = 0$.

E. $f(-1) = 3$.

Question 15

For $y = \sqrt{1 - f(x)}$, $\frac{dy}{dx}$ is equal to

A. $\dfrac{2f'(x)}{\sqrt{1-f(x)}}$

B. $\dfrac{-1}{2\sqrt{1-f'(x)}}$

C. $\dfrac{1}{2}\sqrt{1-f'(x)}$

D. $\dfrac{3}{2(1-f'(x))}$

E. $\dfrac{-f'(x)}{2\sqrt{1-f(x)}}$

Question 16

The inverse of the function $f\colon R^+ \to R,\ f(x) = e^{2x+3}$ is

A. $f^{-1}\colon R^+ \to R \quad f^{-1}(x) = e^{-2x-3}$

B. $f^{-1}\colon R^+ \to R \quad f^{-1}(x) = e^{\frac{x-3}{2}}$

C. $f^{-1}\colon (e^3, \infty) \to R \quad f^{-1}(x) = \log_e\left(\sqrt{x}\right) - \frac{3}{2}$

D. $f^{-1}\colon (e^3, \infty) \to R \quad f^{-1}(x) = e^{\frac{x-3}{2}}$

E. $f^{-1}\colon (e^3, \infty) \to R \quad f^{-1}(x) = -\log_e(2x-3)$

Question 17

The sample space when a fair twelve-sided die is rolled is {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Each outcome is equally likely.

For which one of the following pairs of events are the events independent?

A. {1, 3, 5, 7, 9, 11} and {1, 4, 7, 10}

B. {1, 3, 5, 7, 9, 11} and {2, 4, 6, 8, 10, 12}

C. {4, 8, 12} and {6, 12}

D. {6, 12} and {1, 12}

E. {2, 4, 6, 8, 10, 12} and {1, 2, 3}

Question 18

If $\int_0^a \sec^2(2x)\,dx = \frac{1}{2}$ where $a \in \left(0, \frac{\pi}{4}\right)$ then a is equal to

A. $\frac{\pi}{4}$

B. $\frac{\pi}{8}$

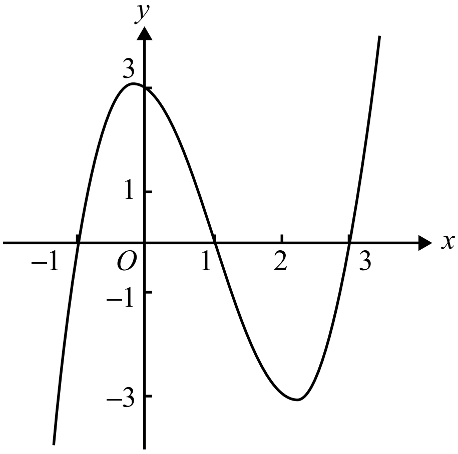
C. $\frac{1}{2}$

D. $\frac{\pi}{12}$

E. $\frac{1}{4}$

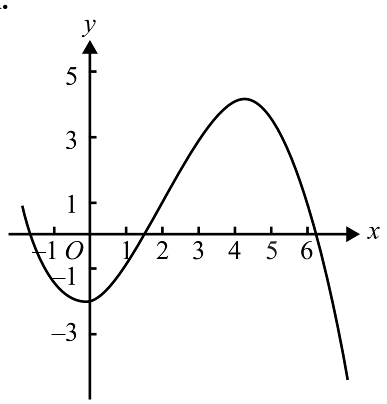
Question 19

The graph of a function f, with domain R, is as shown.

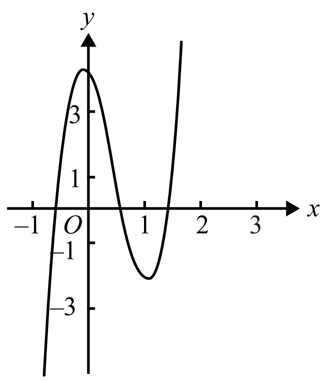


The graph which best represents $1 - f(2x)$ is

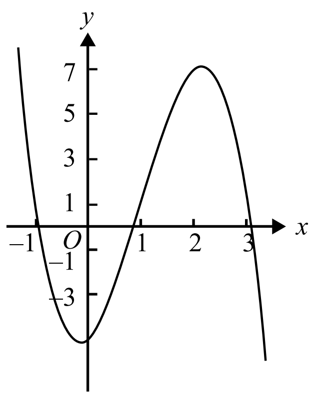
A.



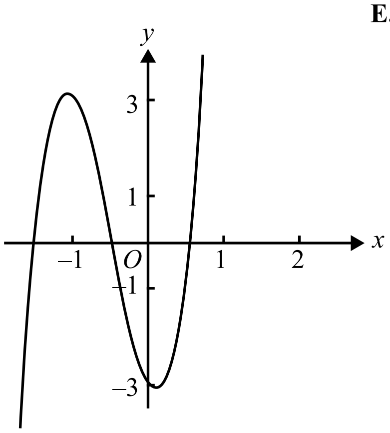
B.



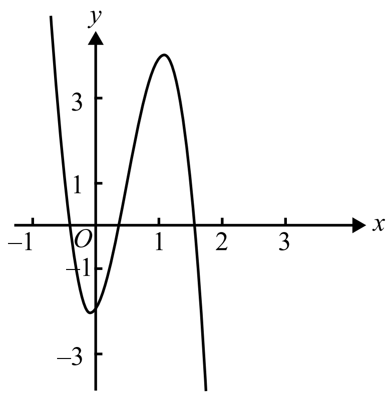
C.



D.



E.



Question 20

The number of solutions for x of the equation $|a\cos(2x)| = |a|$, where $x \in [-2\pi, 2\pi]$ and a is a non-zero constant, is

A. 3

B. 4

C. 5

D. 7

E. 9

Question 21

A cubic function has the rule $y = f(x)$. The graph of the derivative function f' crosses the x-axis at (2, 0) and (–3, 0). The maximum value of the derivative function is 10.

The value of x for which the graph of $y = f(x)$ has a local maximum is

A. –2

B. 2

C. –3

D. 3

E. $-\frac{1}{2}$

Question 22

Consider the region bounded by the x-axis, the y-axis, the line with equation $y = 3$ and the curve with equation $y = \log_e(x-1)$.

The exact value of the area of this region is

A. $e^{-3} - 1$

B. $16 + 3\log_e(2)$

C. $3e^3 - e^{-3} + 2$

D. $e^3 + 2$

E. $3e^2$

END OF SECTION 1

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.

A decimal approximation will not be accepted if an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

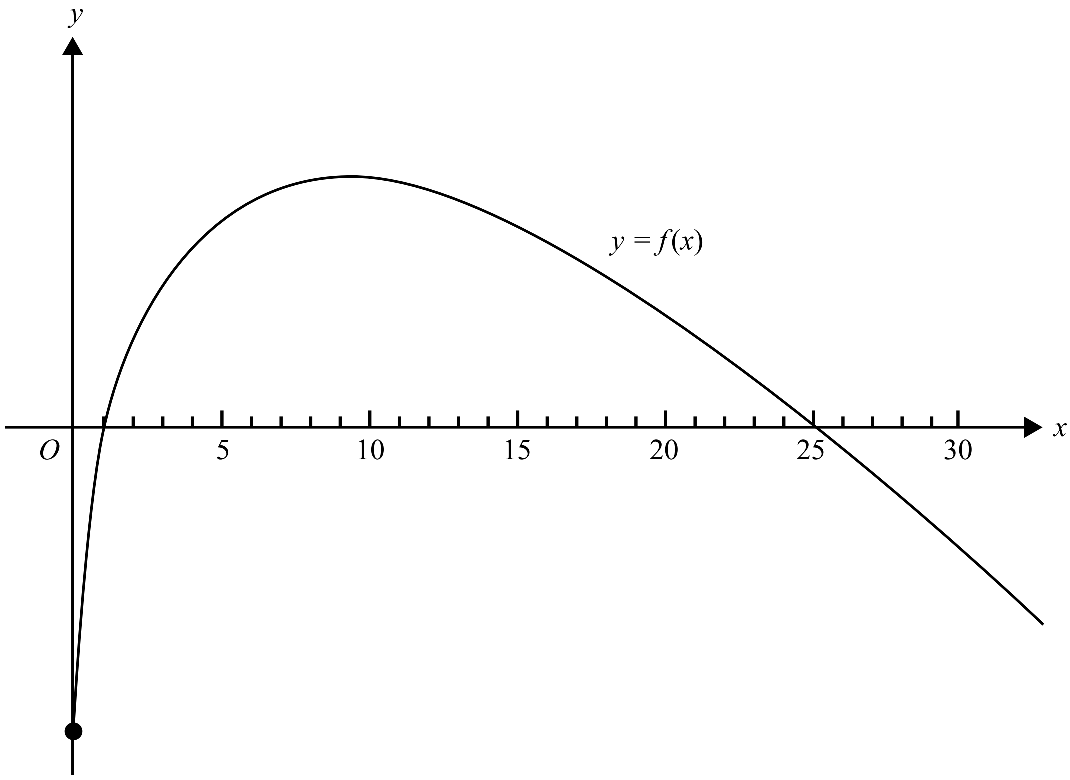
Where an instruction to **use calculus** is stated for a question, you must show an appropriate derivative or anti-derivative.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

Let $f: R^+ \cup \{0\} \rightarrow R,\ f(x) = 6\sqrt{x} - x - 5$.

The graph of $y = f(x)$ is shown below.



a. Use calculus to find the coordinates of the stationary point of f.

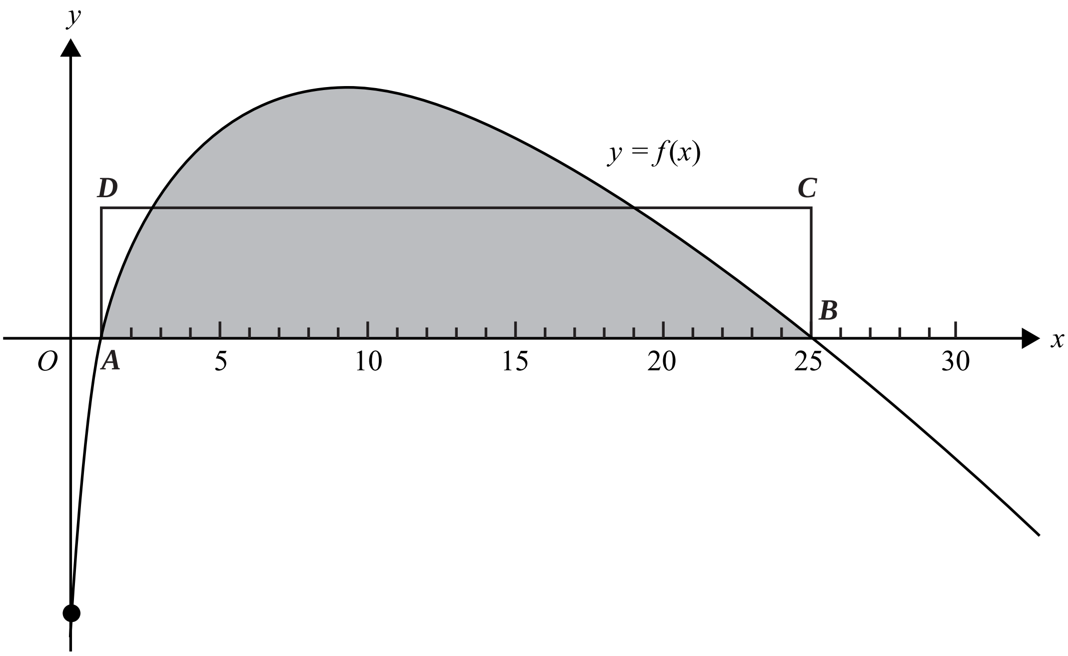
3 marks

b. On the set of axes above, sketch the graph of $y = |f(x)|$.

2 marks

c. Points A and B are the points of intersection of $y = f(x)$ with the x-axis. Point A has coordinates $(1, 0)$ and point B has coordinates $(25, 0)$.

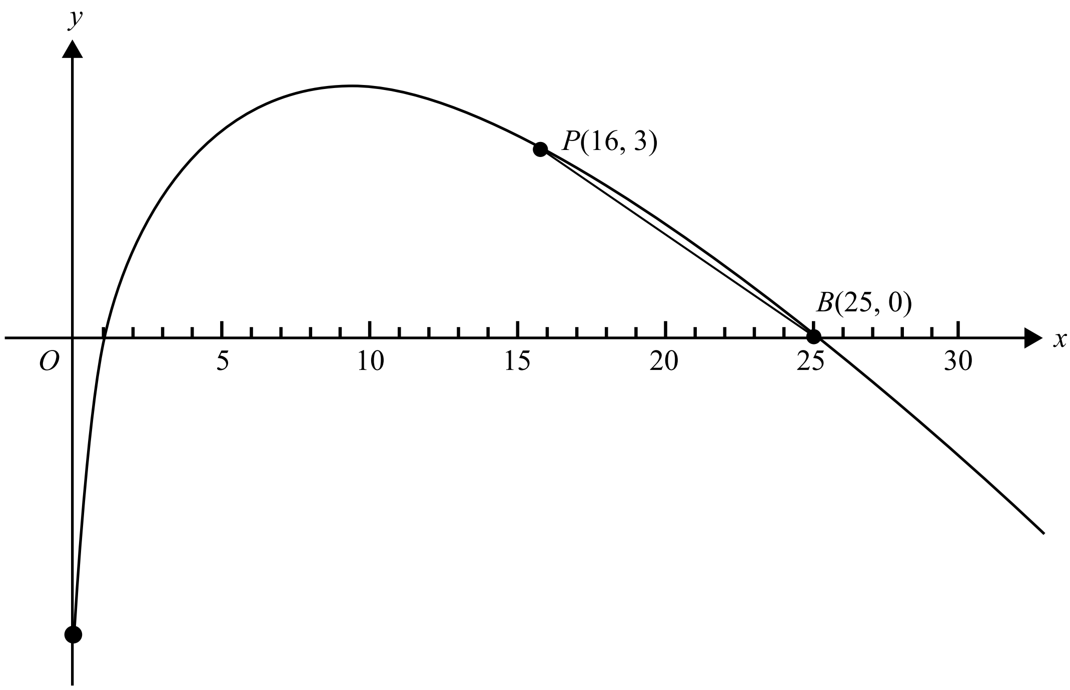
i. Use calculus to find the area of the shaded region.



ii. Hence find the length of AD such that the area of rectangle $ABCD$ is equal to the area of the shaded region.

3 + 1 = 4 marks

d. The points $P(16, 3)$ and $B(25, 0)$ are labelled on the diagram.



i. Find m, the gradient of chord PB. (Exact value to be given.)

ii. Find $a \in [16, 25]$ such that $f'(a) = m$. (Exact value to be given.)

1 + 2 = 3 marks

e. Let $g: R \to R,\ g(x) = x^2$.

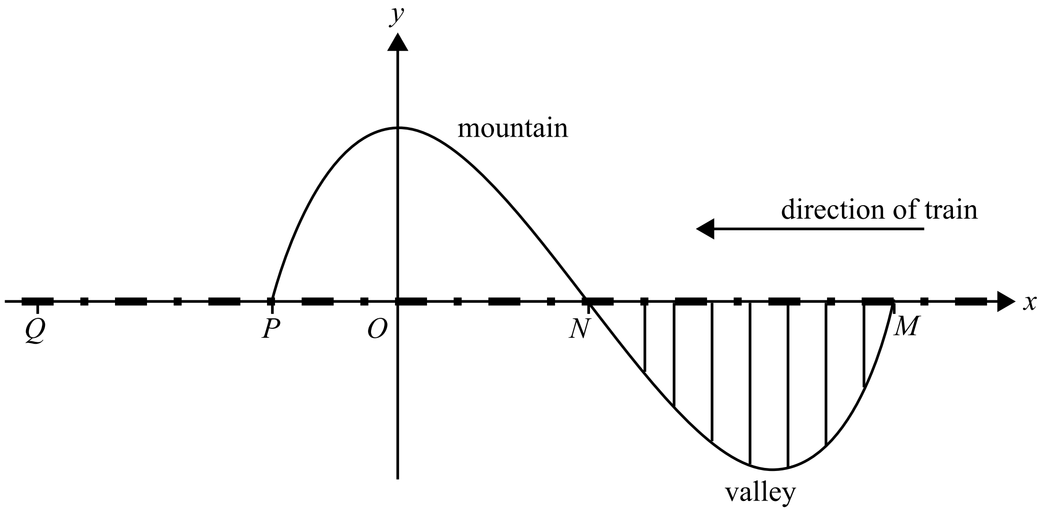
i. Find the rule of $f(g(x))$.

ii. Find the derivative of $f(g(x))$ with respect to x for $x \neq 0$.

2 + 2 = 4 marks

Total 16 marks

Question 2



A train is travelling at 120 km/h along a straight level track from *M* towards *Q*.

The train will travel along a section of track *MNPQ*.

Section *MN* passes along a bridge over a valley.
Section *NP* passes through a tunnel in a mountain.
Section *PQ* is 6.2 km long.

From *M* to *P*, the curve of the valley and the mountain, directly below and above the train track, is modelled by the graph of

$$y = \frac{1}{200}(x^3 + bx^2 + c) \text{ where } b \text{ and } c \text{ are real numbers.}$$

All measurements are in kilometres.

a. The curve from *M* to *P*

- passes through the point *N* (2, 0)
- has a turning point at $x = 4$.

i. From this information write down two simultaneous equations in *b* and *c*.

ii. Hence show that $b = -6$ and $c = 16$.

3 + 2 = 5 marks

b. Find, giving exact values

i. the coordinates of M and P

ii. the maximum depth of the valley below the train track

iii. the maximum height of the mountain above the train track

iv. the length of the tunnel.

3 + 1 + 1 + 1 = 6 marks

The driver sees a large rock on the track at a point Q, 6.2 km from P. The driver puts on the brakes at the instant that the front of the train comes out of the tunnel at P.

From its initial speed of 120 km/h, the train slows down from point P so that its speed v km/h is given by

$$v = k \log_e\left(\frac{(d+1)}{7}\right),$$

where d km is the distance of the front of the train from P and k is a real constant.

c. Find the exact value of k.

1 mark

d. Find the exact distance from the front of the train to the large rock when the train finally stops.

2 marks

Total 14 marks

Question 3

The Bouncy Ball Company (BBC) makes tennis balls whose diameters are normally distributed with mean 67 mm and standard deviation 1 mm. The tennis balls are packed and sold in cylindrical tins that each hold four balls. A tennis ball fits into such a tin if the diameter of the ball is less than 68.5 mm.

a. What is the probability, correct to four decimal places, that a randomly selected tennis ball produced by BBC fits into a tin?

2 marks

BBC management would like each ball produced to have diameter between 65.6 and 68.4 mm.

b. What is the probability, correct to four decimal places, that the diameter of a randomly selected tennis ball made by BBC is in this range?

2 marks

c. **i.** What is the probability, correct to four decimal places, that the diameter of a tennis ball which fits into a tin is between 65.6 and 68.4 mm?

ii. A tin of four balls is selected at random. What is the probability, correct to four decimal places, that at least one of these balls has diameter outside the desired range of 65.6 to 68.4 mm?

1 + 2 = 3 marks

BBC management wants engineers to change the manufacturing process so that 99% of all balls produced have diameter between 65.6 and 68.4 mm. The mean is to stay at 67 mm but the standard deviation is to be changed.

d. What should the new standard deviation be (correct to two decimal places)?

3 marks

BBC sells tennis balls directly to tennis clubs once a year. If a tennis club buys its balls from BBC one year, there is an 80% chance it will buy its balls from BBC the next year. If a tennis club does not buy its balls from BBC one year, there is a 15% chance it will buy its balls from BBC the next year.

Suppose the Melbourne Tennis Club buys its tennis balls from BBC this year.

e. What is the exact probability that it will buy its tennis balls from BBC for the next three years?

2 marks

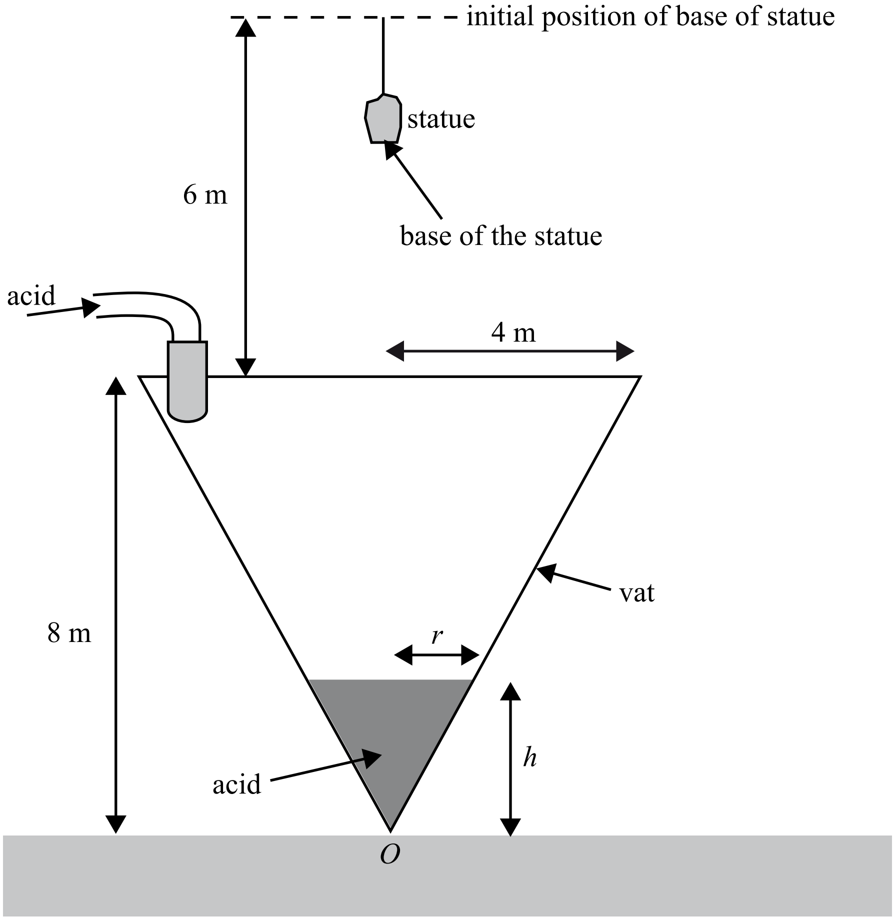
f. What is the exact probability that it will buy its tennis balls from BBC for exactly two of the next three years?

3 marks

Total 15 marks

Question 4

A Zambeji tribe has stolen a precious marble statue from Tasmania Jones. The statue has been tied to a rope and is suspended so that its base is initially 6 metres above the top of a vat. The vat is an inverted right circular cone with base radius 4 metres and height 8 metres.



At 9.00 am the tribe starts to lower the marble statue towards the vat at a rate of 1 metre per hour. At the same time acid begins to be poured into the vat at a constant rate of $\frac{9\pi}{4}$ m^3 per hour. The vat is initially empty. When the statue touches the acid, it will start to dissolve.

At time t hours after 9.00 am, the height of acid in the vat is h metres and the radius of the surface of the acid in the vat is r metres.

a. **i.** Show that $h = 2r$.

ii. Hence find an expression for the volume of acid in the vat at time t, in terms of h.

1 + 1 = 2 marks

b. Show that $\frac{dh}{dt}$, the rate at which the height of the acid is increasing, is $\frac{9}{h^2}$ metres per hour.

2 marks

c. Find, giving exact values

i. the rate at which the height of the acid is increasing when its height is 2 metres

ii. the height of the acid when it is increasing at half the rate found in **c. i.**

1 + 2 = 3 marks

d. **i.** Write an expression for $\frac{dt}{dh}$ in terms of h.

ii. Hence find an expression for the height of the acid in terms of t.

1 + 2 = 3 marks

Tasmania Jones will try to save the statue.

e. **i.** Write an expression for the distance of the base of the statue above ground level t hours after 9.00 am. (The vertex of the cone, O, is at ground level.)

ii. At what time would the statue first touch the acid?

1 + 2 = 3 marks

Total 13 marks

END OF QUESTION AND ANSWER BOOK

Victoria
The Place To Be

MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time.

This formula sheet is provided for your reference.

This page is blank

Mathematical Methods and Mathematical Methods (CAS) Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

$$\int x^n dx=\frac{1}{n+1}x^{n+1}+c, n\neq -1$$

$$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$$

$$\int e^{ax}dx=\frac{1}{a}e^{ax}+c$$

$$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$$

$$\int \frac{1}{x}dx=\log_e|x|+c$$

$$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$$

$$\int \sin(ax)dx=-\frac{1}{a}\cos(ax)+c$$

$$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$$

$$\int \cos(ax)dx=\frac{1}{a}\sin(ax)+c$$

$$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$$

product rule: $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h)\approx f(x)+hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = E(X)$

variance: $\text{var}(X) = \sigma^2 = E((X-\mu)^2) = E(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET