Government of South Australia

SACE Board of SA

External Examination 2012

2012 SPECIALIST MATHEMATICS

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ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator ☐

Brand ____________

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Friday 9 November: 9 a.m.

Time: 3 hours

Pages: 41
Questions: 15

Examination material: one 41-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.
2. This paper consists of three sections:

   Section A (Questions 1 to 9) — 75 marks
   Answer ***all*** questions in Section A.

   Section B (Questions 10 to 13) — 60 marks
   Answer ***all*** questions in Section B.

   Section C (Questions 14 and 15) — 15 marks
   Answer ***one*** question from Section C.
3. Write your answers in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 32 and 40 if you need more space, making sure to label each answer clearly.
4. Appropriate steps of logic and correct answers are required for full marks.
5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)
6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.
7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.
8. Diagrams, where given, are not necessarily drawn to scale.
9. The list of mathematical formulae is on page 41. You may remove the page from this booklet before the examination begins.
10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.
11. Attach your SACE registration number label to the box at the top of this page.

SECTION A (Questions 1 to 9)

(75 marks)

*Answer **all** questions in this section.*

QUESTION 1 (7 marks)

Figure 1 shows a curve similar to the original symbol adopted by the Australian Broadcasting Corporation (ABC).

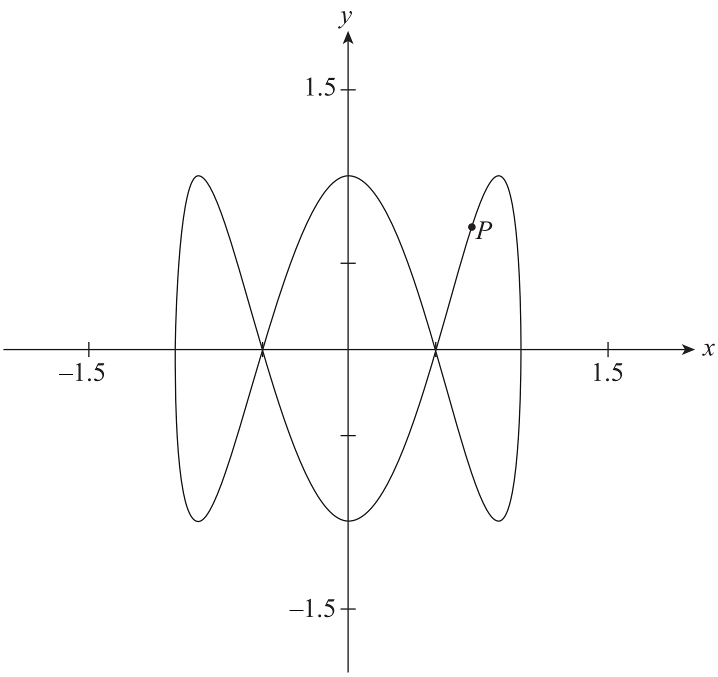

Figure 1

(a) The parametric equations for this curve are

$$\begin{cases} x = \cos t \\ y = \sin 3t \end{cases} \text{ where } 0 \le t \le 2\pi.$$

(i) Find $\frac{dy}{dt}$.

(1 mark)

(ii) Find $\frac{dx}{dt}$.

(1 mark)

(iii) Find $\frac{dy}{dx}$.

(1 mark)

(b) Point P corresponds to the value $t = \frac{\pi}{4}$.

(i) Find the coordinates of P in surd form.

(1 mark)

(ii) Find the slope of the tangent to the curve at P.

(1 mark)

(iii) Find the equation of the tangent to the curve at P.

(2 marks)

QUESTION 2 (10 marks)

(a) Using $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, show that $\dfrac{\mathrm{d}}{\mathrm{d}\theta}(\tan\theta) = \sec^2\theta$.

(2 marks)

(b) Let $f(x) = \sec^2 x \tan^3 x$.

(i) Simplify $\int f(x)\,\mathrm{d}x$.

(2 marks)

(ii) Hence evaluate exactly $\int_0^{\frac{\pi}{3}} \sec^2 x \tan^3 x \,\mathrm{d}x$.

(2 marks)

(iii) Show that $f(x)$ is an odd function.

(2 marks)

(iv) Hence find the value of $\int_{-\frac{\pi}{3}}^{a} f(x)\,\mathrm{d}x$, given that $\int_{0}^{a} f(x)\,\mathrm{d}x = 5$.

(2 marks)

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QUESTION 3 (8 marks)

In Figure 2 points A, B, and C lie on the minor arc of a circle with centre O. The tangents to the circle at A and C meet at point T.

Let $\angle ABC = \beta$ as shown.

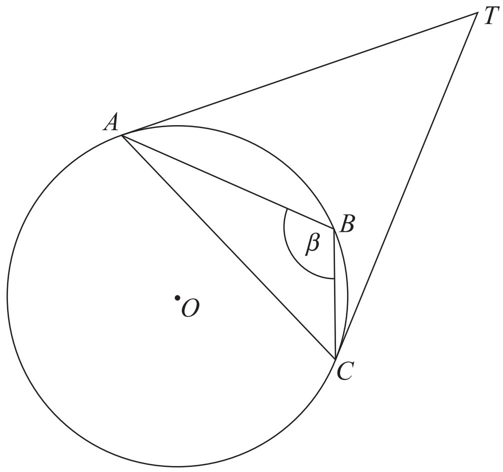

Figure 2

(a) Prove that $\angle ATC = 2\beta - \pi$.

(3 marks)

(b) Prove that quadrilateral $ATCO$ is cyclic.

(1 mark)

(c) Consider the quadrilateral $ABCO$.

(i) For what value of β is $ABCO$ a parallelogram? Give reasons.

(2 marks)

(ii) Prove that AC and OB are perpendicular for the value of β found in part (c)(i).

(2 marks)

QUESTION 4 (9 marks)

Let $\boldsymbol{a} = [1,\ 4,\ -1]$ and $\boldsymbol{b} = [-1,\ 2,\ -2]$.

(a) Find:

(i) $|\boldsymbol{b}|$.

(1 mark)

(ii) $\boldsymbol{a} \bullet \boldsymbol{b}$.

(1 mark)

(b) Show clearly that the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$, shown as θ in Figure 3, is $\frac{\pi}{4}$.

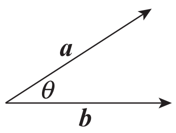

Figure 3

(2 marks)

(c) Given point $A(3,\ -4,\ 2)$, as shown in Figure 4, find two points M and N so that $\angle MAN = \frac{\pi}{4}$.

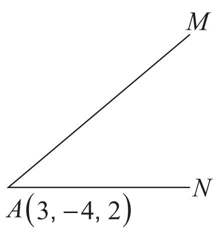

Figure 4

(2 marks)

(d) Show that the vectors $\boldsymbol{c} = [4,\ 1,\ -1]$ and $\boldsymbol{d} = [2,\ -1,\ -2]$ are at an angle of $\frac{\pi}{4}$.

(1 mark)

(e) State two vectors, neither of which is parallel to any of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, or $\boldsymbol{d}$, that are at an angle of $\frac{\pi}{4}$.

(2 marks)

QUESTION 5 (5 marks)

Let $f(x) = x^3 + ax + b$, where a and b are constants.

(a) Given that $(x+1)$ is a factor of $f(x)$, show that $b - a = 1$.

(2 marks)

(b) When $f(x)$ is divided by $(x+2)$, the remainder is 6.

Show that $b - 2a = 14$.

(1 mark)

(c) Find a and b, and hence write $f(x)$ as a product of linear factors.

(2 marks)

QUESTION 6 (9 marks)

Consider the quadratic iteration $z \to z^2 + c,\ \ z_0 = 0,\ \ c = 0.3 + 0.5i$.

(a) Investigate the long-term behaviour of this iteration and complete the table below.

n	z_n
1	
2	
3	
4	
5	
6	

(4 marks)

(b) Describe the apparent long-term behaviour of this iteration.

(1 mark)

(c) Round your final six z values from part (a) to two decimal places in the table below.

z_n

(2 marks)

(d) Plot the rounded values from part (c) on the diagram of the Mandelbrot set shown in Figure 5.

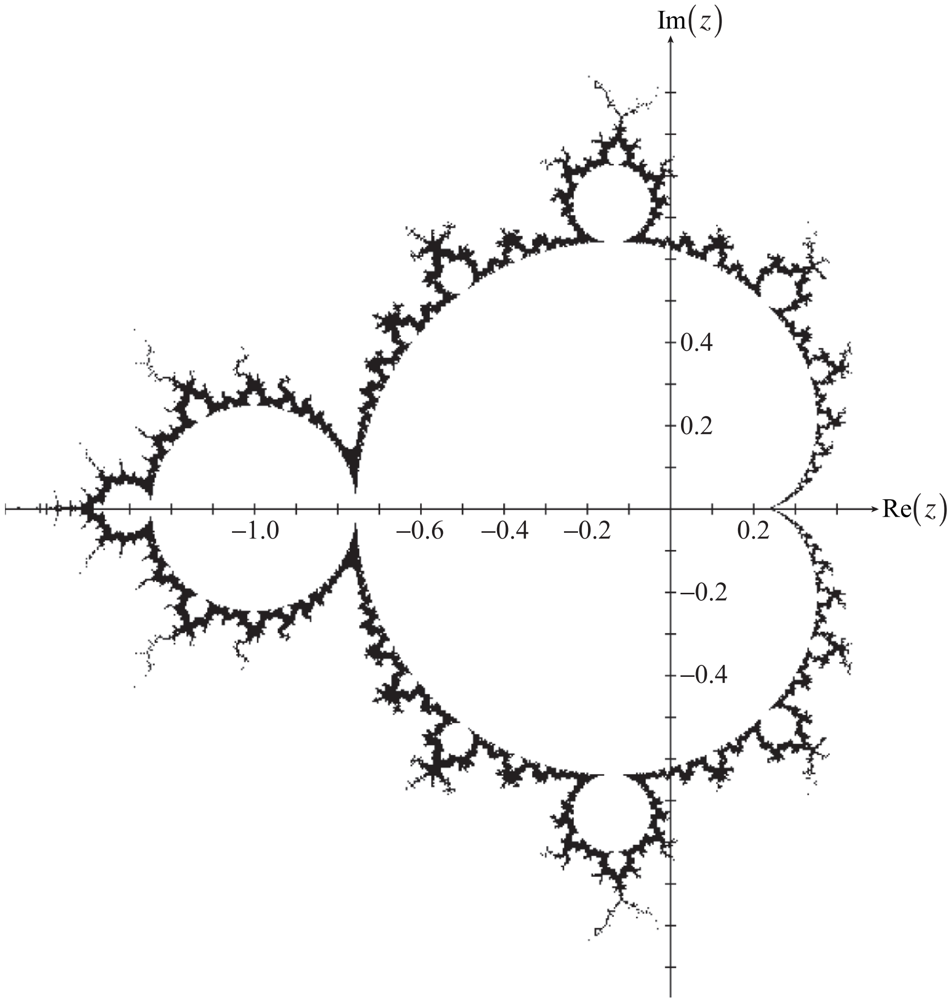

Figure 5

(2 marks)

QUESTION 7 (8 marks)

A particle P is moving around an ellipse so that its position $P(t)$ at time t is given by the parametric equations

$$\begin{cases} x(t) = 4\cos t \\ y(t) = 2\sin t \end{cases} \text{ where } 0 \le t \le 2\pi.$$

(a) Sketch this ellipse on the axes in Figure 6.

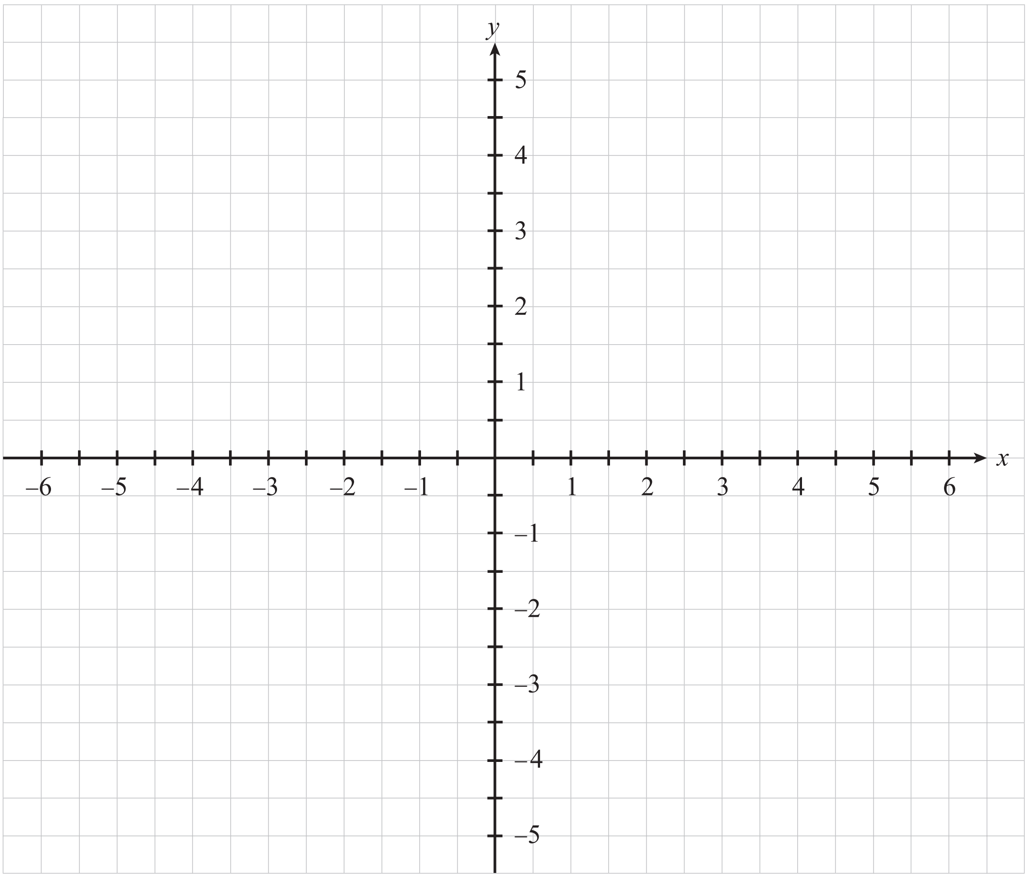

Figure 6 (2 marks)

(b) Point $P(x, y)$ is in the first quadrant and lies on the ellipse.

Point A has coordinates $(2, 0)$.

Show that the distance $S(t)$, between $P(x, y)$ and $A(2, 0)$, is given by

$$S(t) = \sqrt{x^2 - 4x + 4 + y^2}.$$

(2 marks)

(c) Show that

$$\frac{dS}{dt} = \frac{1}{S}\left((x-2)\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

(2 marks)

(d) Find the rate at which the distance between $P(x, y)$ and $A(2, 0)$ is changing when $t = \frac{\pi}{4}$.

(2 marks)

QUESTION 8 (9 marks)

Consider the 2×2 matrix $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$.

(a) (i) Show that $A^2=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{bmatrix}$.

(3 marks)

(ii) Hence show that $A^3=\begin{bmatrix}\cos 3\theta & \sin 3\theta\\ -\sin 3\theta & \cos 3\theta\end{bmatrix}$.

(2 marks)

(b) Use an inductive argument to prove that

$$A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

for any positive integer n.

(3 marks)

(c) If $\theta = \dfrac{\pi}{20}$, find the smallest positive integer value of n for which $A^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(1 mark)

QUESTION 9 (10 marks)

Figure 7 shows a circle in the complex plane.

The circle passes through the origin and the labelled points A and B on the real and imaginary axes.

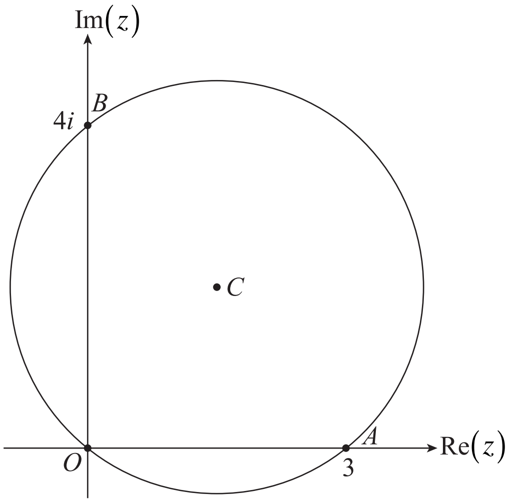

Figure 7

(a) Prove that AB is a diameter of the circle.

(1 mark)

(b) Find the complex number at point C, the centre of the circle.

(2 marks)

(c) Find the radius of the circle.

(1 mark)

(d) Write an equation that describes all points z on the circle.

(3 marks)

(e) Find the length of the circle's minor arc OB.

(3 marks)

SECTION B (Questions 10 to 13)

(60 marks)

*Answer **all** questions in this section.*

QUESTION 10 (15 marks)

$A(3, 1, 3)$, $B(0, 2, 6)$, and $C(6, -6, -9)$ are three points in space.

(a) Find:

(i) $\overrightarrow{AB}$.

(1 mark)

(ii) $\overrightarrow{AB} \times \overrightarrow{BC}$.

(2 marks)

(iii) the equation of the plane P containing A, B, and C.

(2 marks)

(b) (i) Find the equation of line l_1 that passes through point $D(14, -20, -20)$ and is parallel to $\overrightarrow{AB}$.

(2 marks)

(ii) Find the equation of line l_2 that passes through $D(14, -20, -20)$ and is normal to the plane P.

(1 mark)

(iii) Show that l_1 and l_2 are perpendicular.

(1 mark)

(iv) Find where l_2 intersects the plane P. Let this point be E.

(2 marks)

(v) Show that B, C, and E are collinear.

(1 mark)

(c) Looking at triangle ABC in the direction of the normal to the plane P, identify which one of the following diagrams correctly represents the position of line l_1 in relation to the triangle.

Explain your answer, with reference to your work in part (b).

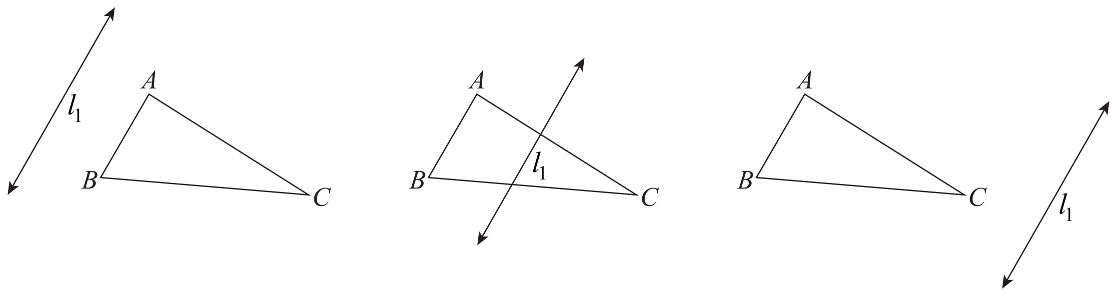

Diagram 1 **Diagram 2** **Diagram 3**

(3 marks)

QUESTION 11 (15 marks)

(a) Show that $2\sin\left(2x+\frac{2\pi}{3}\right)=\sqrt{3}\cos 2x-\sin 2x$.

(2 marks)

(b) (i) On the axes in Figure 8, sketch the graph of $y=e^{\frac{-x}{3}}\left(\sqrt{3}\cos 2x-\sin 2x\right)$ for $0\le x\le \pi$.

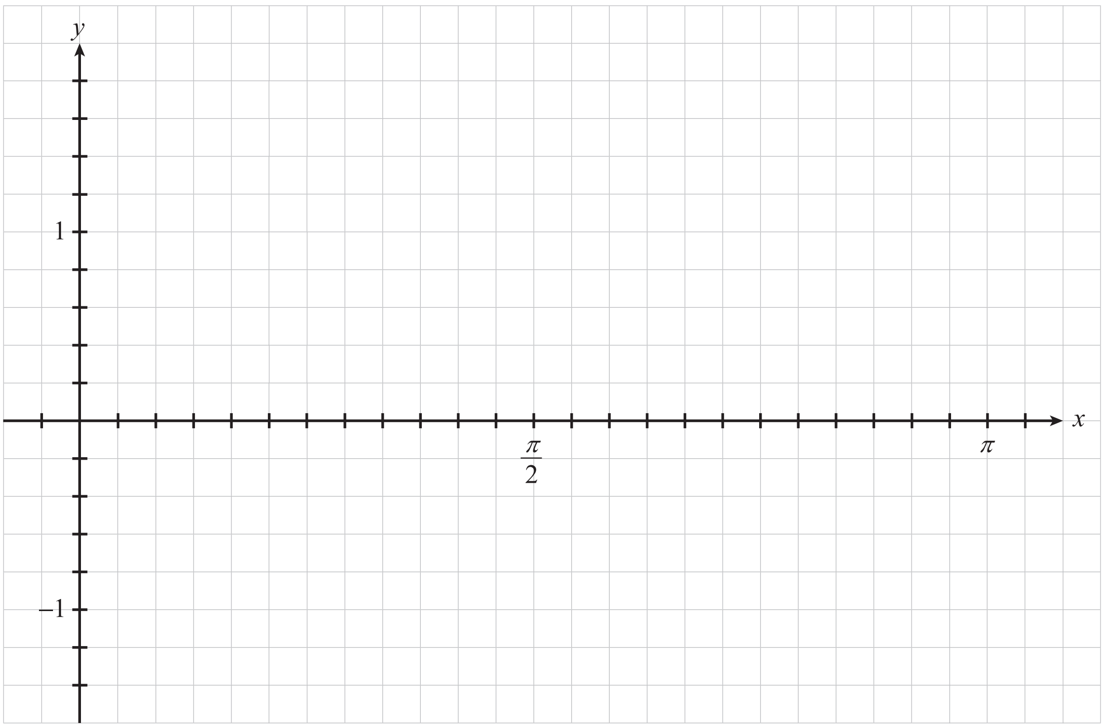

Figure 8

(3 marks)

(ii) Find the exact values of the x-intercepts for $0 \le x \le \pi$.

(2 marks)

(c) The equation $s = 2e^{\frac{-t}{3}} \sin\left(2t + \frac{2\pi}{3}\right)$ represents the position of a particle that is oscillating with a decreasing speed.

Show that:

(i) the particle's velocity is given by $v = 2e^{\frac{-t}{3}}\left(-\frac{1}{3}\sin\left(2t + \frac{2\pi}{3}\right) + 2\cos\left(2t + \frac{2\pi}{3}\right)\right)$.

(1 mark)

(ii) the particle's acceleration can be expressed as $a = -\frac{2}{3}v - \frac{37}{9}s$.

(3 marks)

(d) For those times when $s = 0$, find:

(i) all values of v for $0 \le t \le \pi$.

(2 marks)

(ii) all values of a for $0 \le t \le \pi$.

(2 marks)

QUESTION 12 (15 marks)

This photograph of a Native bush rat cannot be reproduced here for copyright reasons

Native bush rat

Source: G. Lewis, http://museumvictoria.com.au

The Australian native bush rat, *Rattus fuscipes*, and the introduced feral black rat, *Rattus rattus*, compete for common resources in bushland. The Bindiai Scrub Nature Reserve Council has commissioned population studies of the rats living in the nature reserve to determine whether or not pest control of the feral black rat is needed. The findings of the studies are shown in the following table.

This photograph of a Feral black rat cannot be reproduced here for copyright reasons

Feral black rat

Source: I. McCann, http://museumvictoria.com.au

Bindiai Scrub Nature Reserve Rat Population Estimates

Rat Type	Current Population	Growth
Feral black rat	$x(0)=210$	$x'=-x+3y$
Native bush rat	$y(0)=700$	$y'=2x-6y$

The system of growth equations given in the table is known to have a solution of the form

$$\begin{cases} x(t)=A+Be^{-7t} \\ y(t)=C+De^{-7t} \end{cases}$$

where A, B, C, and D are constants.

(a) (i) Use the given form of $x(t)$ to find $x'(t)$.

(1 mark)

(ii) Hence find values for A and B using the initial conditions.

(3 marks)

(b) Hence, or otherwise, find $y(t)$.

(2 marks)

(c) On Figure 9, draw the graphs of $x(t)$ and $y(t)$ on the same axes.

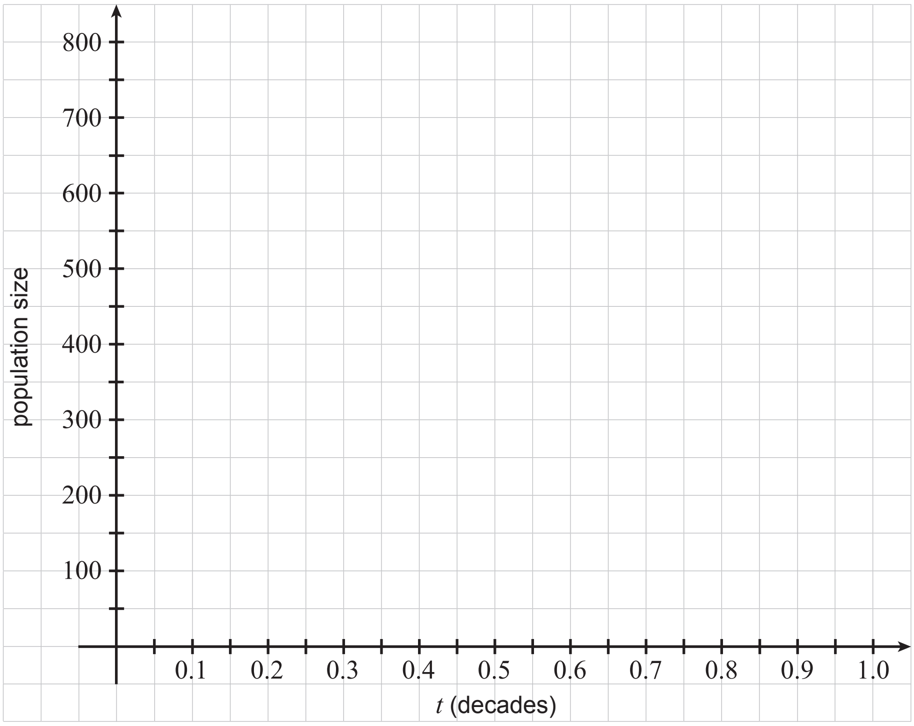

Figure 9

(4 marks)

(d) At what time are the rat populations the same size?

(1 mark)

(e) Describe the behaviour of the rat populations as $t \to \infty$.

(2 marks)

(f) What recommendations would you make to the Bindiai Scrub Nature Reserve Council? Justify your recommendations.

(2 marks)

QUESTION 13 (15 marks)

(a) (i) Write $\sqrt{2} - i\sqrt{2}$ in $r\text{cis}\theta$ form, where $r \geq 0$.

(2 marks)

(ii) Hence find $\left(\sqrt{2} - i\sqrt{2}\right)^4$.

(1 mark)

(b) Solve $z^4 = -16$, writing your answers exactly in $r\text{cis}\theta$ form.

(3 marks)

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(c) Show that $\dfrac{z^7 + z^4 + 16z^3 + 16}{z^3 + 1} = z^4 + 16$.

(1 mark)

(d) Use your results from parts (b) and (c) to solve the equation

$$z^7 + z^4 + 16z^3 + 16 = 0$$

writing your answers exactly in $r\text{cis}\theta$ form.

(3 marks)

(e) Plot your solutions from part (d) on the Argand diagram in Figure 10, labelling them $z_1, z_2, \ldots, z_7$ anticlockwise from the smallest positive argument.

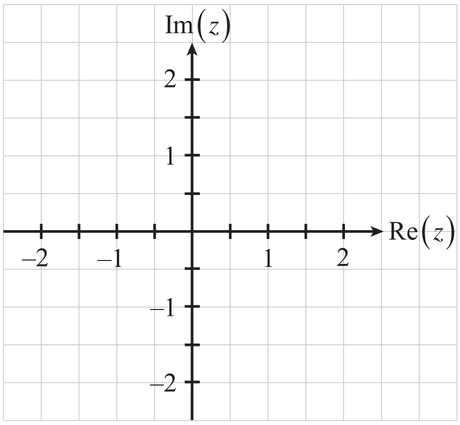

Figure 10

(2 marks)

(f) (i) Find $|z_1|+|z_2|+|z_3|+\ldots+|z_7|$.

(1 mark)

(ii) Explain why $|z_1|+|z_2|+|z_3|+\ldots+|z_7| \geq |z_1+z_2+z_3+\ldots+z_7|$.

(1 mark)

(iii) Find $|z_1+z_2+z_3+\ldots+z_7|$.

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 12(a)(ii) continued').

SECTION C (Questions 14 and 15)

(15 marks)

*Answer **one** question from this section, **either** Question 14 **or** Question 15.*

*Answer **either** Question 14 **or** Question 15.*

QUESTION 14 (15 marks)

Source: © Ichip/Dreamstime.com

A skydiver jumps from an aeroplane and free-falls before opening his parachute.

The speed v of the skydiver t seconds after he opens his parachute can be modelled by the differential equation

$$\frac{dv}{dt} = -0.413\left(v^2 - k^2\right)$$

where k is a positive constant that is related to the type of parachute, the mass of the skydiver, and gravity.

(a) Verify that $\dfrac{1}{v^2 - k^2} = \dfrac{1}{2k}\left(\dfrac{1}{v-k} - \dfrac{1}{v+k}\right)$.

(1 mark)

(b) Hence show that $\displaystyle\int \frac{1}{v^2 - k^2}\,dv = \frac{1}{2k}\ln\left|\frac{v-k}{v+k}\right| + c$, where c is a constant of integration.

(2 marks)

(c) Using the results of parts (a) and (b), show that the differential equation has a solution

$$v(t) = k\frac{\left(1 + Ae^{-0.826kt}\right)}{\left(1 - Ae^{-0.826kt}\right)}$$

where A is a constant.

(3 marks)

(d) Let the parachute open at $t = 0$ and let $v(0) = 10$ metres per second.

(i) Find the constant A in terms of k.

(2 marks)

For a particular skydiver, $k^2 = 23.7$.

(ii) Find $v(t)$, the speed of this skydiver, solely in terms of t.

(3 marks)

(iii) Find the limiting speed of the skydiver.

(1 mark)

(iv) Graph $v(t)$ on the axes in Figure 11 for the first 1 second after the parachute opens, clearly showing the information found in part (d)(iii).

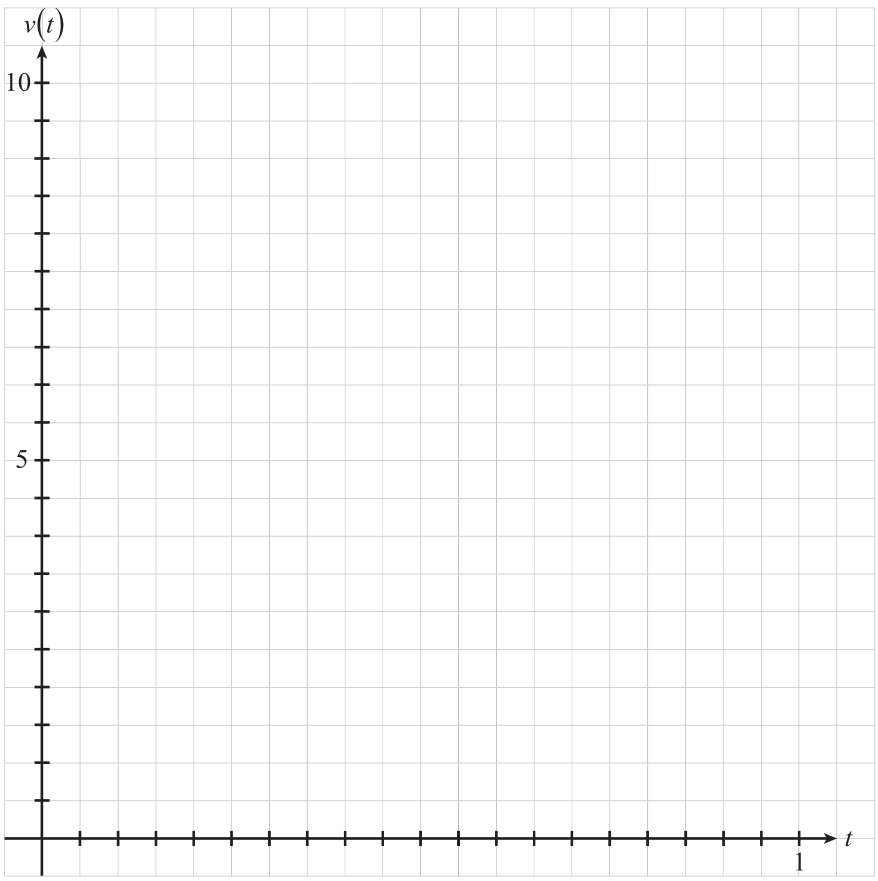

Figure 11

(3 marks)

*Answer **either** Question 14 **or** Question 15.*

QUESTION 15 (15 marks)

$A(1, -1, 2)$, $B(3, 2, 7)$, and $C(5, 6, 6)$ are three points in space, as shown in Figure 12, and t is a parameter such that $0 \le t \le 1$.

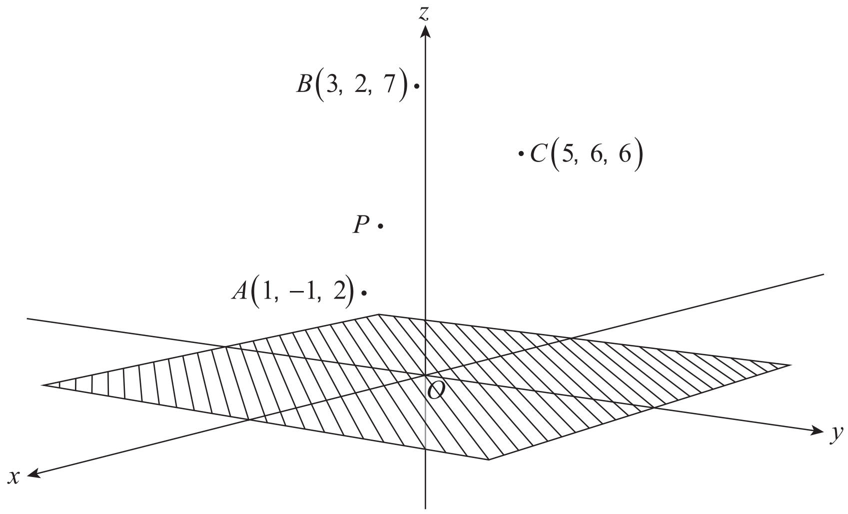

Figure 12

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

(1 mark)

(b) P is a point on $\overrightarrow{AB}$ such that $\overrightarrow{AP} = t\overrightarrow{AB}$, where $0 \le t \le 1$.

Draw $\overrightarrow{OA}$, $\overrightarrow{AP}$, and $\overrightarrow{OP}$ on Figure 12 and show that $\overrightarrow{OP} = [1+2t,\ 3t-1,\ 2+5t]$.

(3 marks)

(c) Q is a point on $\overrightarrow{BC}$ such that $\overrightarrow{BQ} = t\overrightarrow{BC}$, where $0 \le t \le 1$.

Show that $\overrightarrow{OQ} = [3 + 2t,\ 2 + 4t,\ 7 - t]$.

(1 mark)

(d) R is a point on $\overrightarrow{PQ}$ such that $\overrightarrow{PR} = t\overrightarrow{PQ}$, where $0 \le t \le 1$.

Show that $\overrightarrow{OR} = \left[4t + 1,\ t^2 + 6t - 1,\ 2 + 10t - 6t^2\right]$.

(3 marks)

(e) Calculate the position of R for $t = 0$, $t = \frac{1}{3}$, $t = \frac{1}{2}$, and $t = 1$.

(2 marks)

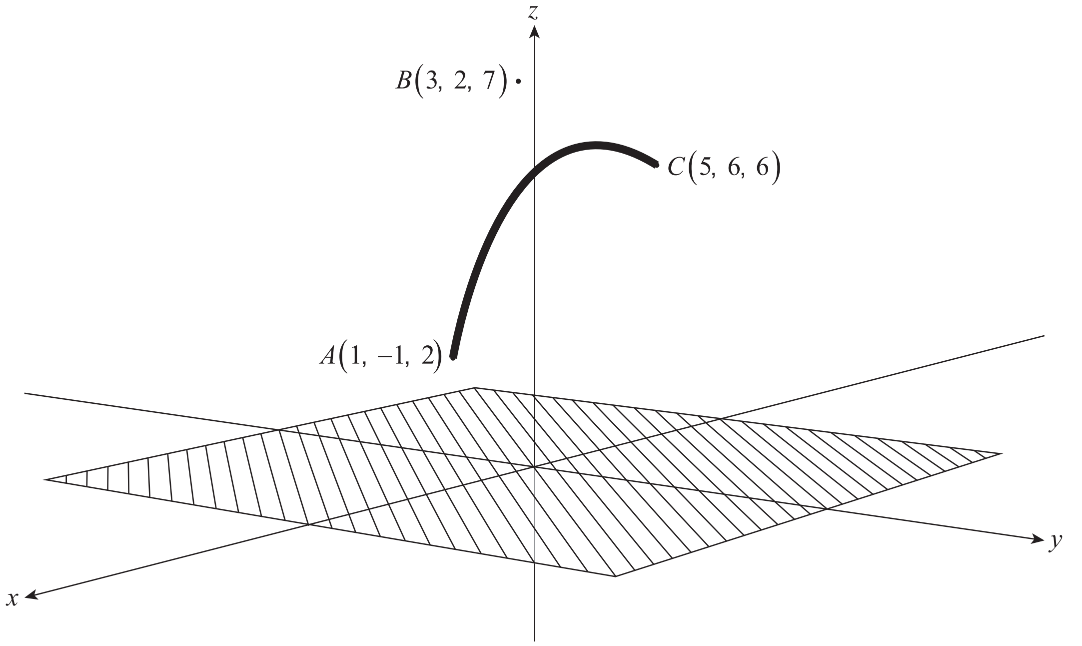

Figure 13

(f) Figure 13 shows the path of R as it travels from A to C.

Find the maximum height, above the XOY plane, of the path travelled by R.

(2 marks)

(g) The length of a three-dimensional curved distance along a path defined by

$(x(t),\ y(t),\ z(t))$ between $t = a$ and $t = b$ is given by $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$.

Find the length of the curved path travelled by R from A to C.

(3 marks)

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You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 12(a)(ii) continued').

You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular Functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \tfrac{1}{2}(A \pm B) \cos \tfrac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Properties of Derivatives

$\dfrac{d}{dx}\{f(x)\,g(x)\} = f'(x)\,g(x) + f(x)\,g'(x)$

$\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f(g(x)) = f'(g(x))\,g'(x)$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a Point to a Plane

The distance from (x_1, y_1, z_1) to $Ax + By + Cz + D = 0$ is given by

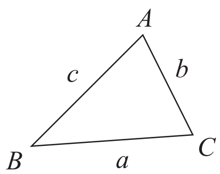
$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$

Mensuration

Area of sector $= \tfrac{1}{2} r^2 \theta$

Arc length $= r\theta$
(where θ is in radians)

In any triangle ABC:



Area of triangle $= \tfrac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$