Government of South Australia

SACE Board of SA

External Examination 2011

2011 SPECIALIST MATHEMATICS

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ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator ☐

Brand ______

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Friday 11 November: 9 a.m.

Time: 3 hours

Pages: 41
Questions: 16

Examination material: one 41-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. This paper consists of three sections:

   Section A (Questions 1 to 10) 75 marks
   Answer ***all*** questions in Section A.

   Section B (Questions 11 to 14) 60 marks
   Answer ***all*** questions in Section B.

   Section C (Questions 15 and 16) 15 marks
   Answer ***one*** question from Section C.

3. Write your answers in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 19, 23, 27, 31, and 40 if you need more space, making sure to label each answer clearly.

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 41. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

SECTION A (Questions 1 to 10)

(75 marks)

*Answer **all** questions in this section.*

QUESTION 1 (6 marks)

Consider the curve with parametric equations

$$x(t)=e^{t},\ y(t)=t-2t^{4}.$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(3 marks)

(b) Hence show that the tangent to the curve when $x=1$ has equation $y=x-1$.

(3 marks)

QUESTION 2 (7 marks)

Figure 1 shows points $A(0, -3, 1)$, $B(3, 2, 2)$, and $P(4, -1, 3)$ in three-dimensional space.

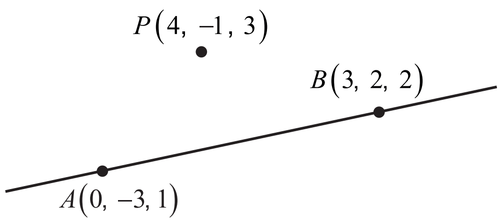

Figure 1

(a) Find $\overrightarrow{AB}$.

(1 mark)

(b) Find the area of triangle APB.

(4 marks)

(c) Find the coordinates of a point C on line AB such that the area of triangle APC is twice the area of triangle APB.

(2 marks)

QUESTION 3 (4 marks)

Figure 2 shows points B and C on a semicircle with centre O and diameter AD.

Given that $\angle BAC = \alpha$, $\angle ABC = \beta$, and $\angle OAB = \gamma$, prove that $\alpha + \beta + \gamma = \dfrac{\pi}{2}$.

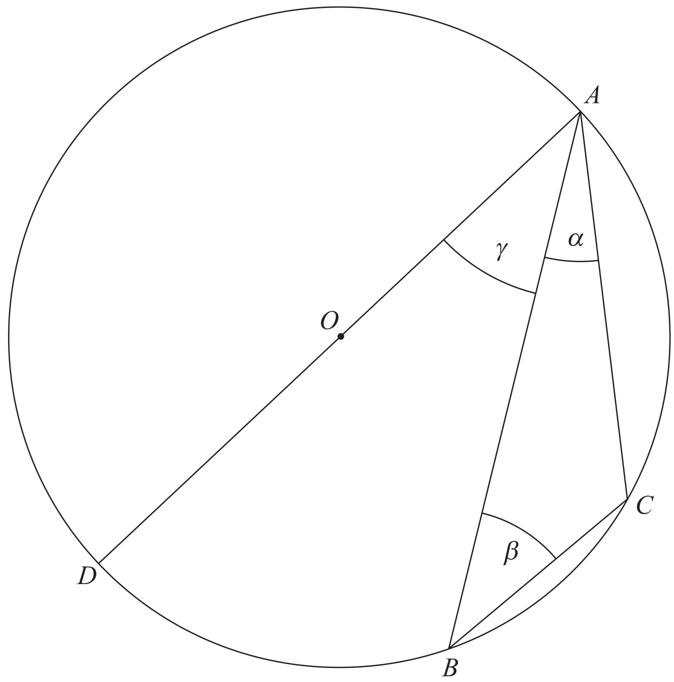

Figure 2

(4 marks)

QUESTION 4 (8 marks)

(a) Draw the graph of $f(x) = \sqrt[3]{\sin x}$ on the axes provided in Figure 3.

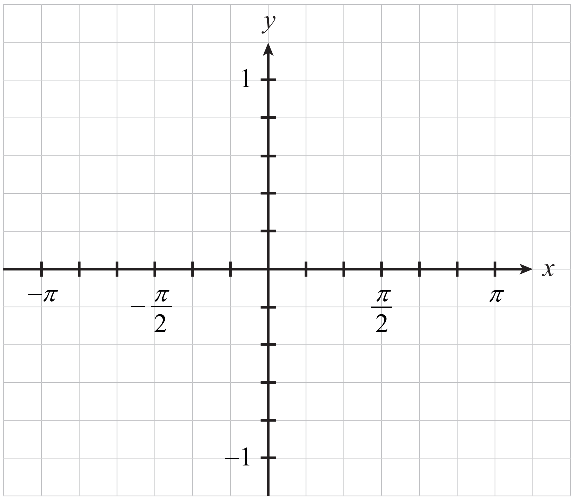

Figure 3 (3 marks)

(b) Prove that $f(x)$ is an odd function.

(2 marks)

(c) On your graph in part (a), shade the region represented by $\int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x}\, dx$. (1 mark)

(d) Hence evaluate $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt[3]{\sin x}\, dx$, explaining your answer.

(2 marks)

QUESTION 5 (7 marks)

A mechanised conveyor belt delivers mined salt in conical piles.

As the salt is delivered, the piles can be modelled as a cone that remains twice as wide at the base as it is high (see Figure 4).

The conveyor belt delivers salt at a rate of 30 cubic metres per minute. The volume of a cone with height h and base radius r is given by

$$V = \frac{1}{3}\pi r^2 h.$$

Source: www.istockphoto.com

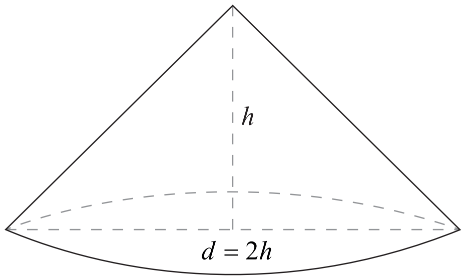

Figure 4

(a) Show that $\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}$.

(3 marks)

(b) Find the rate at which diameter d is changing when the salt pile is 10 metres high.

(4 marks)

QUESTION 6 (8 marks)

Consider the polynomial $P(x)=\left(a^2+1\right)x\left(x-2\right)^n+2a$, where a is a real constant.

(a) What values can n take?

(1 mark)

(b) Show that if $x-1$ is a factor of $P(x)$, then a can take two values. Find these values.

(3 marks)

(c) Find values for the constants a and n if $P(x)$ has a factor of $x-1$ and a remainder of -164 on division by $x+1$.

(4 marks)

QUESTION 7 (7 marks)

(a) Complete the table below by adding the appropriate squared number.

n	Terms	Sum of Terms
1	9	9
2	$9+36$	45
3	$9+36+81$	126
4	$9+36+81+144$	270
5	$9+36+81+144+\ldots$	

(1 mark)

(b) Use the sum of 5 terms to calculate the sum of 6 terms, continuing the pattern.

(1 mark)

(c) Find a cubic polynomial model that will calculate the sum of n terms, continuing the pattern.

(1 mark)

(d) Show that your cubic polynomial model from part (c) can be written in the form

$$S_n = \frac{3n(2n+1)(n+1)}{2}.$$

(1 mark)

(e) Hence, or otherwise, give an inductive argument to justify that

$$9+36+\ldots+(3n)^2 = \frac{3n(2n+1)(n+1)}{2}.$$

(3 marks)

QUESTION 8 (7 marks)

Consider the quadratic iteration $z \to z^2 + c,\ z_0 = 0,\ c = -\frac{1}{8} - \frac{3}{4}i.$

(a) Investigate the long-term behaviour of this iteration and complete the table below.

n	z_n
1	
2	
3	
4	

(3 marks)

(b) Clearly describe the apparent long-term behaviour of this iteration.

(1 mark)

(c) Use $z_{n+1} = z_n^2 + c$ to show that the equation for three-cycle behaviour is

$$z = \left(\left(z^2 + c\right)^2 + c\right)^2 + c.$$

(2 marks)

(d) Use your answers to parts (a) and (c) to explain why it is not surprising that there are no complex conjugate pairs in the completed table in part (a).

(1 mark)

QUESTION 9 (9 marks)

Early mobile phones did not have the capacity to take photographs. When new phones with a camera were introduced, their market share was initially very small. However, the market share of these new phones grew very rapidly, and old phones without a camera became almost obsolete.

Source: Yuris, www.dreamstime.com

Let x denote the market share of the new phones, where $0 < x < 1$, at time t months after their introduction. The market share of the old phones is then $1 - x$, and the growth of the new phones' market share can be modelled by the differential equation

$$\frac{dx}{dt} = 0.8x(1-x).$$

(a) Show that $\dfrac{1}{x(1-x)} = \dfrac{1}{x} + \dfrac{1}{1-x}$.

(1 mark)

(b) Market research indicated that the initial market share of the new phones was $\frac{1}{10}\%$.

Solve the differential equation to show that $x = \dfrac{1}{1+999e^{-0.8t}}$.

(5 marks)

(c) Find when the market share of the new phones was growing most rapidly.

(2 marks)

(d) After how many months were 90% of all phones of the new type?

(1 mark)

QUESTION 10 (12 marks)

(a) (i) If $u = r\text{cis}\theta$ and $v = R\text{cis}\phi$, explain why $|u+v| \le r+R$.

(1 mark)

(ii) When does equality hold?

(1 mark)

Figure 5 shows a circular region S that is centred at the origin and includes the boundary, situated in the complex plane.

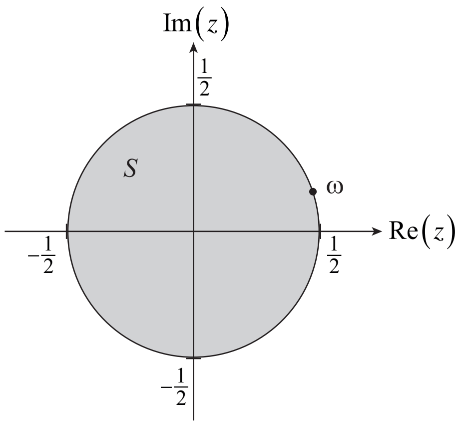

Figure 5

(b) (i) Write an inequality describing precisely the complex numbers z in the region S.

(1 mark)

(ii) Indicate on Figure 5 the position of ω^2, where ω is the number shown. (2 marks)

(c) Consider the quadratic iteration $z \to z^2 + \frac{i}{2}$, where z_0 is in the region S.

(i) Use the triangle inequality to show that $|z_1| \le \frac{3}{4}$.

(2 marks)

(ii) Calculate the exact value of $|z_1|$ when $z_0 = \frac{1}{2}$.

(2 marks)

(iii) Using the result of part (a)(ii), find a complex number z_0 for which $|z_1| = \frac{3}{4}$.

(3 marks)

PLEASE TURN OVER

SECTION B (Questions 11 to 14)

(60 marks)

*Answer **all** questions in this section.*

QUESTION 11 (15 marks)

(a) Line l_1 passes through points $A(4, -2, 4)$ and $B(6, -4, 5)$.

Find the equation of line l_1 in parametric form.

(2 marks)

(b) Line l_2 has Cartesian equations $x-3=\frac{y-2}{2}=\frac{z-5}{2}$.

Show that l_1 and l_2 intersect at point $C(2, 0, 3)$.

(3 marks)

(c) (i) Find $[2, -2, 1] \times [1, 2, 2]$.

(2 marks)

(ii) Show that the equation of the plane containing lines l_1 and l_2 is $2x + y - 2z = -2$.

(2 marks)

(d) (i) Show that the plane $x + z = 5$ is perpendicular to the plane $2x + y - 2z = -2$.

(2 marks)

PLEASE TURN OVER

(ii) Find the coordinates of point D on the plane $x+z=5$ such that D is closest to $B(6, -4, 5)$.

(2 marks)

(iii) Find the coordinates of point E such that BE is bisected by D.

Show that E is on the plane $2x+y-2z=-2$.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 10(c)(ii) continued').

QUESTION 12 (16 marks)

(a) (i) Show that $\dfrac{z^6 + 3z^4 + 3z^2 + 2}{z^2 + 2} = z^4 + z^2 + 1$.

(2 marks)

(ii) Find the purely imaginary zeros of $z^6 + 3z^4 + 3z^2 + 2$.

(1 mark)

(b) (i) Solve $z^3 = -1$, giving answers in $r\text{cis}\theta$ form.

(3 marks)

(ii) Using your results from part (b)(i), solve $\left(z^2+1\right)^3 = -1$, showing clearly that

$$z = \pm\sqrt{2}\,i,\ \text{cis}\frac{\pi}{3},\ \text{cis}\frac{2\pi}{3},\ \text{cis}\frac{-2\pi}{3},\ \text{cis}\frac{-\pi}{3}.$$

(4 marks)

(c) (i) Let the solutions of $\left(z^2+1\right)^3=-1$ be z_1, z_2, z_3, z_4, z_5, and z_6, with z_1 in the first quadrant and arguments increasing anticlockwise from the positive real axis.

Graph the solutions on the Argand diagram in Figure 6 and label them as z_1, z_2, z_3, z_4, z_5, and z_6.

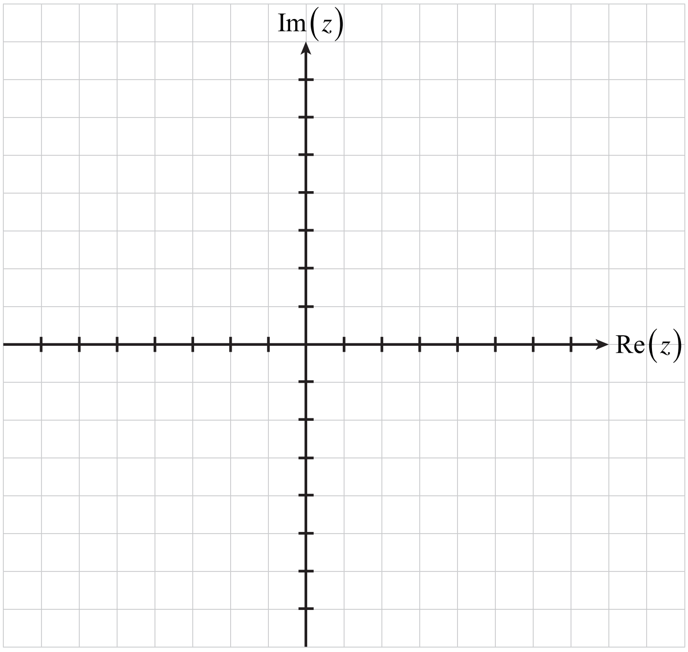

Figure 6

(2 marks)

(ii) Show that $|z_1-z_6|=\sqrt{3}$.

(2 marks)

(iii) Find $|z_1-z_6|+|z_2-z_5|+|z_3-z_4|$.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 10(c)(ii) continued').

QUESTION 13 (15 marks)

The position of a moving particle can be modelled by the differential system

$$\begin{cases} x' = 3x - 2y \\ y' = 2x - y \end{cases}$$ with initial conditions $x(0) = -2,\ y(0) = -3.$

(a) Find the particle's initial velocity and speed.

(2 marks)

(b) The differential system has a solution of the form $\begin{cases} x(t) = (A + Bt)e^{t} \\ y(t) = (C + Dt)e^{t}. \end{cases}$

(i) Use the given form for $x(t)$ to find $x'(t)$.

(2 marks)

(ii) Hence find values for A and B, using the initial conditions.

(3 marks)

(c) Hence, or otherwise, find $y(t)$.

(2 marks)

(d) Draw the path of the particle on the slope field in Figure 7.

Mark on this path the position and velocity of the particle at $t=0$ and $t=1$.

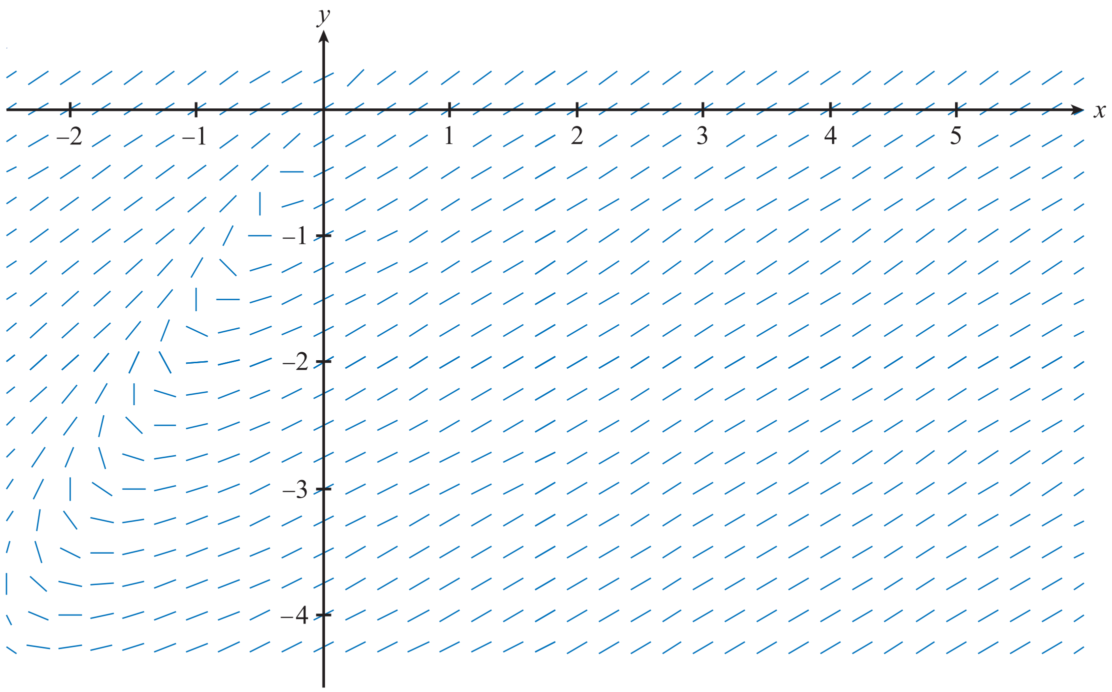

Figure 7

(5 marks)

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(e) Describe the particle's position as $t \to \infty$.

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 10(c)(ii) continued').

QUESTION 14 (14 marks)

A downhill snow skier moves in a path with her position defined by

$$x(t) = t^2 + 4t$$
$$y(t) = 5\sin 2t$$

where $0 \le t \le 2\pi$ is in seconds and x and y are in metres.

In Figure 8, the x-axis may be considered as the centre line down the slope that the skier is on.

This photograph cannot be reproduced here for copyright reasons.

Source: Adapted from www.wallpaperbase.com

(a) (i) Use the equations given above to graph the path of the skier on the axes in Figure 8.

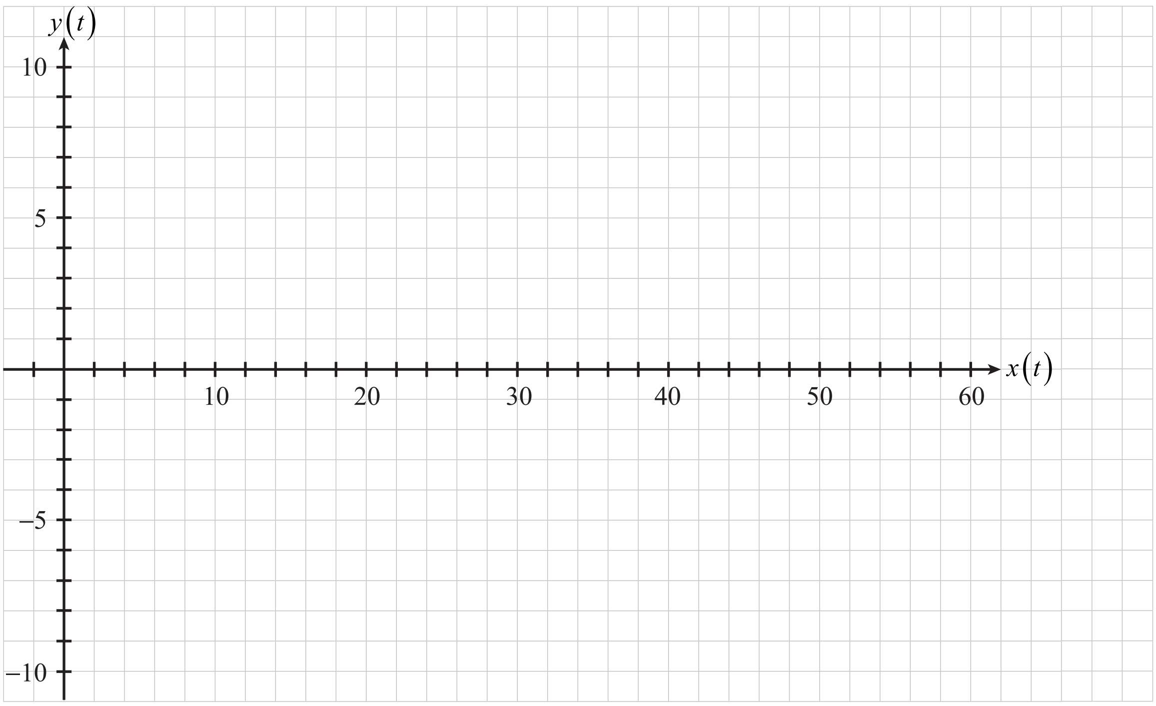


Figure 8

(3 marks)

(ii) Find the exact times when the skier crosses the centre line.

(2 marks)

(iii) Find exactly the first time when the skier is furthest away from the centre line.

(1 mark)

(b) Find the velocity vector of the skier at time t seconds.

(2 marks)

(c) (i) Show that $s(t)$, the skier's speed at time t seconds, is given by

$$s(t)=\sqrt{(2t+4)^2+100\cos^2 2t}.$$

(1 mark)

(ii) Find the maximum speed of the skier and the time at which it occurs.

(2 marks)

(d) The skier contests a 200-metre downhill race. She travels so that her position is defined by

$$\begin{aligned} x(t) &= t^2 + 4t \\ y(t) &= 5\sin 2t \end{aligned} \quad \text{where } t \geq 0.$$

(i) How long does it take the skier to travel 200 metres downhill?

(1 mark)

(ii) How many times after starting does she cross the centre line?

(1 mark)

(iii) What is her speed when she crosses the centre line for the last time?

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 10(c)(ii) continued').

SECTION C (Questions 15 and 16)

(15 marks)

*Answer **one** question from this section, **either** Question 15 **or** Question 16.*

Answer ***either*** *Question 15* ***or*** *Question 16.*

QUESTION 15 (15 marks)

Source: Mrallen, www.dreamstime.com

In Malaysian paddy fields, barn owls have been introduced in an attempt to control rats, which are pests that damage the rice crop.

The following differential system models the rates of change of the populations R of rats and B of barn owls, with time t measured in months:

$$\frac{\mathrm{d}R}{\mathrm{d}t} = 0.516R - 0.0172RB$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = 0.00145RB - 2.175B.$$

This photograph cannot be reproduced here for copyright reasons.

Source: G.R. Singleton, www.knowledgebank.irri.org

(a) (i) Rewrite the differential equation for the rate of change of the rat population if there are no barn owls ($B = 0$).

(1 mark)

(ii) Solve the differential equation from part (a)(i) for an initial rat population of R_0.

(2 marks)

(iii) Comment on the growth of the rat population in the absence of barn owls.

(1 mark)

(b) When the rats support a constant population of barn owls and the barn owls keep the rats under control, the system is said to be in *equilibrium*. Neither population is changing.

(i) Explain why the populations at equilibrium can be found by solving the equations

$$0.516R - 0.0172RB = 0$$
$$0.00145RB - 2.175B = 0.$$

(1 mark)

(ii) One solution to these equations is $R = 0,\ B = 0$.

Find the other solution (the equilibrium solution).

(2 marks)

(c) Figure 9 shows the slope field for $\begin{cases} \dfrac{dR}{dt} = 0.516R - 0.0172RB \\ \dfrac{dB}{dt} = 0.00145RB - 2.175B \end{cases}$ with two solution curves drawn.

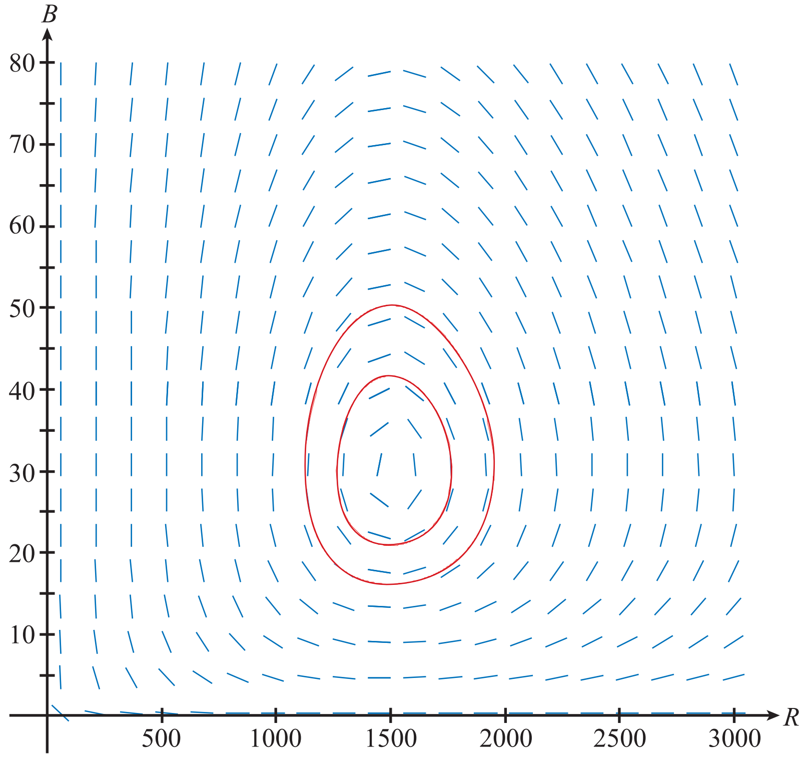


Figure 9

(i) Give an expression for $\frac{dB}{dR}$.

(1 mark)

(ii) On the slope field in Figure 9, plot the equilibrium point (R, B) found in part (b)(ii). (1 mark)

(d) Consider initial conditions of 1500 rats and 10 barn owls.

(i) Draw a solution curve on the slope field in Figure 9. (1 mark)

(ii) Find the value of $\frac{dR}{dt}$.

(1 mark)

(iii) Hence indicate with an arrow on your solution curve what happens to the rat and barn owl populations. (1 mark)

(iv) Use your solution curve to give an estimate for the maximum value of the rat population.

(1 mark)

(v) Briefly describe the long-term interactive behaviour of the rat and barn owl populations.

(2 marks)

*Answer **either** Question 15 **or** Question 16.*

QUESTION 16 (15 marks)

For the online game of ‘stick cricket’, when a player hits a six, the ball leaves a partial red trail across the screen. This is shown in Figure 10.

This screen capture cannot be reproduced here for copyright reasons.

Figure 10

Adding a set of Cartesian axes to the screen allows the partial red trail to be analysed by using a Bézier curve. This is shown in Figure 11.

This screen capture cannot be reproduced here for copyright reasons.

Figure 11

Source for Figures 10 and 11: www.sticksports.com (adapted for Figure 11)

The parametric equations of the black Bézier curve shown in Figure 11 are

$$x(t) = 2t^2 - 16t + 4$$
$$y(t) = -6t^2 + 8t + 2$$
where $0 \le t \le 1$.

(a) (i) Find the coordinates of the curve's initial point A and final point C.

(1 mark)

(ii) Find H, the highest point on the curve.

(2 marks)

(b) (i) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ and write it in the form $a + \frac{b}{t+c}$, where a, b, and c are constants.

(2 marks)

(ii) Hence find the steepest point on the curve.

(1 mark)

(c) Figure 11 shows that a control point for the black Bézier curve is $B(-4, 6)$.

(i) With the aid of a diagram, explain why $AC < AH + HC < AB + BC$.

(2 marks)

(ii) Using your answers from parts (a)(i) and (ii), calculate exact values for AC, $AH + HC$, and $AB + BC$.

(2 marks)

(iii) Let l be the length along the Bézier curve.

Write l, AC, $AH+HC$, and $AB+BC$ in ascending order.

(1 mark)

(d) The length, l, along a parametric curve $(x(t),\ y(t))$ between $t=r$ and $t=s$ is given by

$$l=\int_{t=r}^{t=s}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t.$$

Calculate the length along the Bézier curve and hence verify your placement of l in part (c)(iii).

(2 marks)

(e) Clearly the Bézier curve stops before the edge of the screen but the ball would disappear at the edge of the screen. Assume that the model continues past the value $t=1$ and the ball 'travels' until $y(t)=0$.

Find the corresponding value of t and the total length along the curve for this value of t.

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 10(c)(ii) continued').

You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 SPECIALIST MATHEMATICS

Circular Functions

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1$

$\quad = 1 - 2\sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

$\sin A \pm \sin B = 2 \sin \tfrac{1}{2}(A \pm B) \cos \tfrac{1}{2}(A \mp B)$

$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$

$\cos A - \cos B = -2 \sin \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)$

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$A^{-1} = \dfrac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Derivatives

$f(x) = y$	$f'(x) = \dfrac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
$\ln x = \log_e x$	$\dfrac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

Properties of Derivatives

$\dfrac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$

$\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

$\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Distance from a Point to a Plane

The distance from (x_1, y_1, z_1) to

$Ax + By + Cz + D = 0$ is given by

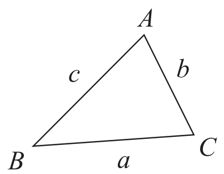
$\dfrac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Mensuration

Area of sector $= \tfrac{1}{2} r^2 \theta$

Arc length $= r\theta$
(where θ is in radians)

In any triangle ABC:



Area of triangle $= \tfrac{1}{2} ab \sin C$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

$a^2 = b^2 + c^2 - 2bc \cos A$

SPECIALIST MATHEMATICS 2011

ACKNOWLEDGMENT

The photograph for Question 5 on page 6 is É iStockphoto.com/Ngataringa.

The photograph for Question 9 on page 12 is É Yuris/Dreamstime.com.

The photograph of the owl for Question 15 on page 12 is É Mrallen/Dreamstime.com.