Government of South Australia

SACE Board of SA

External Examination 2012

2012 MATHEMATICAL STUDIES

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ATTACH SACE REGISTRATION NUMBER LABEL TO THIS BOX

Graphics calculator ☐

Brand ______________

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Tuesday 30 October: 1.30 p.m.

Time: 3 hours

Pages: 41
Questions: 16

Examination material: one 41-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. Answer ***all*** parts of Questions 1 to 16 in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 7, 33, 38, and 39 if you need more space, making sure to label each answer clearly.

3. The total mark is approximately 141. The allocation of marks is shown below:

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Marks	7	5	6	6	6	8	5	11	6	8	11	8	12	13	12	17

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 41. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

QUESTION 1

(a) Find the derivative of the following functions. There is no need to simplify your answers.

(i) $f(x) = (6x - 1)^{10}$.

(2 marks)

(ii) $y = \dfrac{5x^4}{1 + 3e^{-2x}}$.

(3 marks)

(b) For the matrix $A = \begin{bmatrix} p & 4 \\ -2 & 0 \end{bmatrix}$, write down the matrix A^{-1}.

(2 marks)

QUESTION 2

Consider the matrix $X = \begin{bmatrix} a & 1 & a^2 \\ 1 & 0 & 1 \\ 3a & 1 & a \end{bmatrix}$, where a is a real number.

(a) Find $|X|$.

(3 marks)

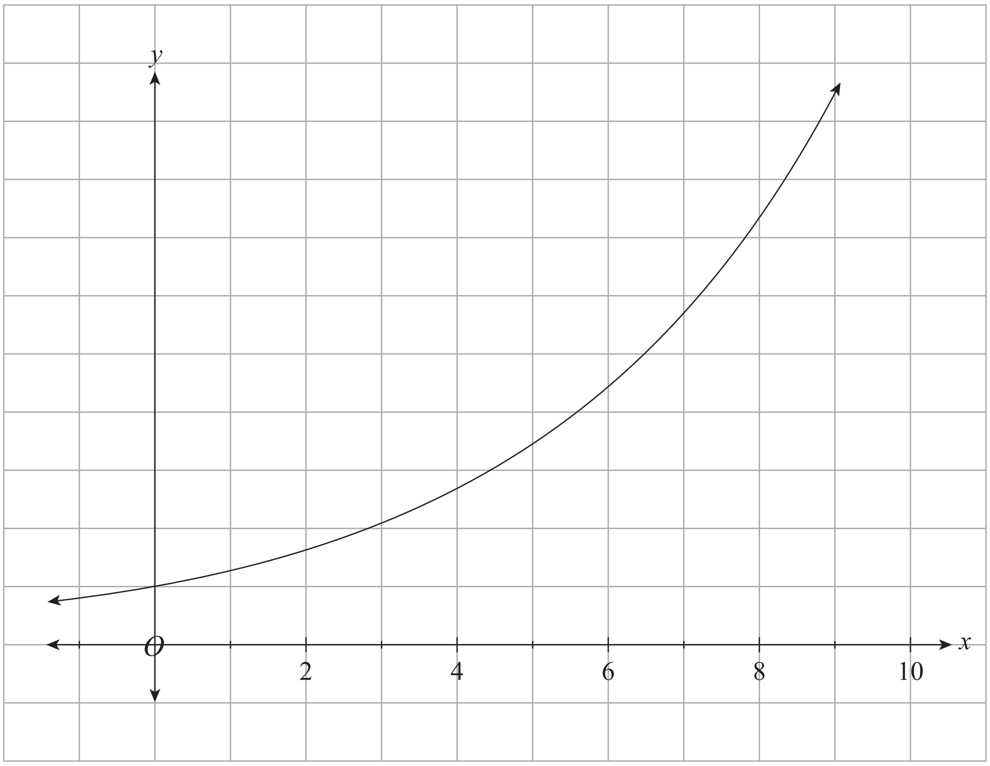
(b) Find the values of a such that X^{-1} exists.

(2 marks)

PLEASE TURN OVER

QUESTION 3

Consider the function $f(x)=2e^{0.25x}$, which is graphed below:



Let L_n represent an underestimate of the area between the graph of $y=f(x)$ and the x-axis for $0 \le x \le 8$, calculated using n rectangles of equal width.

(a) If two rectangles of equal width are used, the underestimate is $L_2 = 29.75$, to two decimal places.

On the graph above, draw the two rectangles of equal width used to obtain this underestimate.

(1 mark)

(b) Find L_4, to two decimal places.

(3 marks)

If eight rectangles of equal width are used, $L_8 = 44.99$, to two decimal places.

(c) Calculate $\int_0^8 2e^{0.25x}\,dx$, to two decimal places.

(1 mark)

(d) Comment on the relationship between L_n and $\int_0^8 2e^{0.25x}\,dx$.

(1 mark)

QUESTION 4

Consider the following matrices, A, B, and C, where x is a real number:

$$A = \begin{bmatrix} 2 & x & 0 \\ -1 & 0 & 7 \end{bmatrix},\ B = \begin{bmatrix} 4 & 3 & -2 \\ -1 & 0 & 3 \end{bmatrix},\text{ and } C = \begin{bmatrix} 0 & -2 \\ -3 & 6 \\ 1 & 0 \end{bmatrix}.$$

(a) Calculate $2A - B$.

(2 marks)

(b) Calculate AC.

(2 marks)

(c) Explain, with specific reference to the dimensions of matrices A and B, why the product AB cannot be calculated.

(1 mark)

(d) Write down a matrix D such that the product DB can be calculated.

(1 mark)

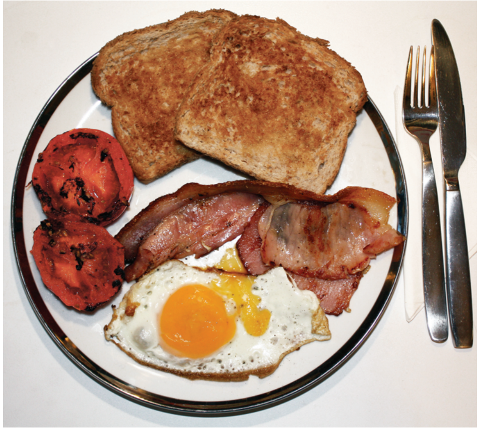
You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 1(a)(ii) continued').

PLEASE TURN OVER

QUESTION 5

A cafe has four meals on its breakfast menu. The meals consist of combinations of four breakfast components: toast, egg, bacon, and fried tomato.

- *Full Breakfast $9.50*: two slices of toast, one egg, one serving of bacon, and one fried tomato
- *Big Breakfast $16.00*: three slices of toast, two eggs, two servings of bacon, and one fried tomato
- *Vegetarian Breakfast $10.40*: two slices of toast, two eggs, and two fried tomatoes
- *Bacon and Eggs $9.90*: two slices of toast, two eggs, and one serving of bacon.



The price of each meal is equal to the sum of the price of the breakfast components of which it consists.

(a) Represent the information above as a system of linear equations, using clearly defined variables.

(3 marks)

(b) Hence determine the price of each breakfast component.

(2 marks)

(c) The cafe decides to add a fifth meal to its breakfast menu:

- Gluten-free Breakfast: two eggs, one serving of bacon, and two fried tomatoes.

Using your answer to part (b), determine the price of the Gluten-free Breakfast.

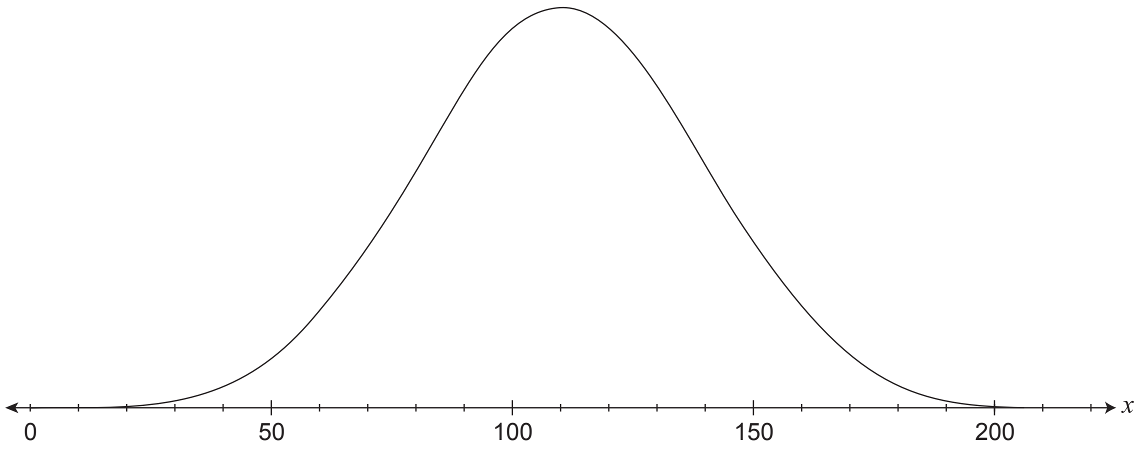
(1 mark)

QUESTION 6

A bank has an investment that has an initial worth of \$100 million. The worth of this investment varies over time. Let X represent the worth of this investment in a year's time. X can be modelled by a normal distribution with a mean of $\mu = \$110$ million and a standard deviation of $\sigma = \$30$ million.

(a) This normal distribution is graphed below.

On the x-axis, clearly show the position of μ, $\mu + 3\sigma$, and $\mu - 3\sigma$.



(1 mark)

(b) Using the mathematical model above, find the probability that this investment will be worth \$60 million or less in a year's time.

(2 marks)

The first percentile of X is the value of a such that $P(X \leq a) = 0.01$.

(c) Find the first percentile for this investment.

(1 mark)

The bank has a second investment that has an initial worth of \$100 million. The worth of this investment in a year's time can be modelled by a normal distribution with a mean of $\mu =$ \$110 million and a standard deviation of σ.

(d) Given that the first percentile for this investment is \$25 million, find the value of σ.

(3 marks)

(e) Hence, what can be said about this second investment in comparison with the investment described on page 10?

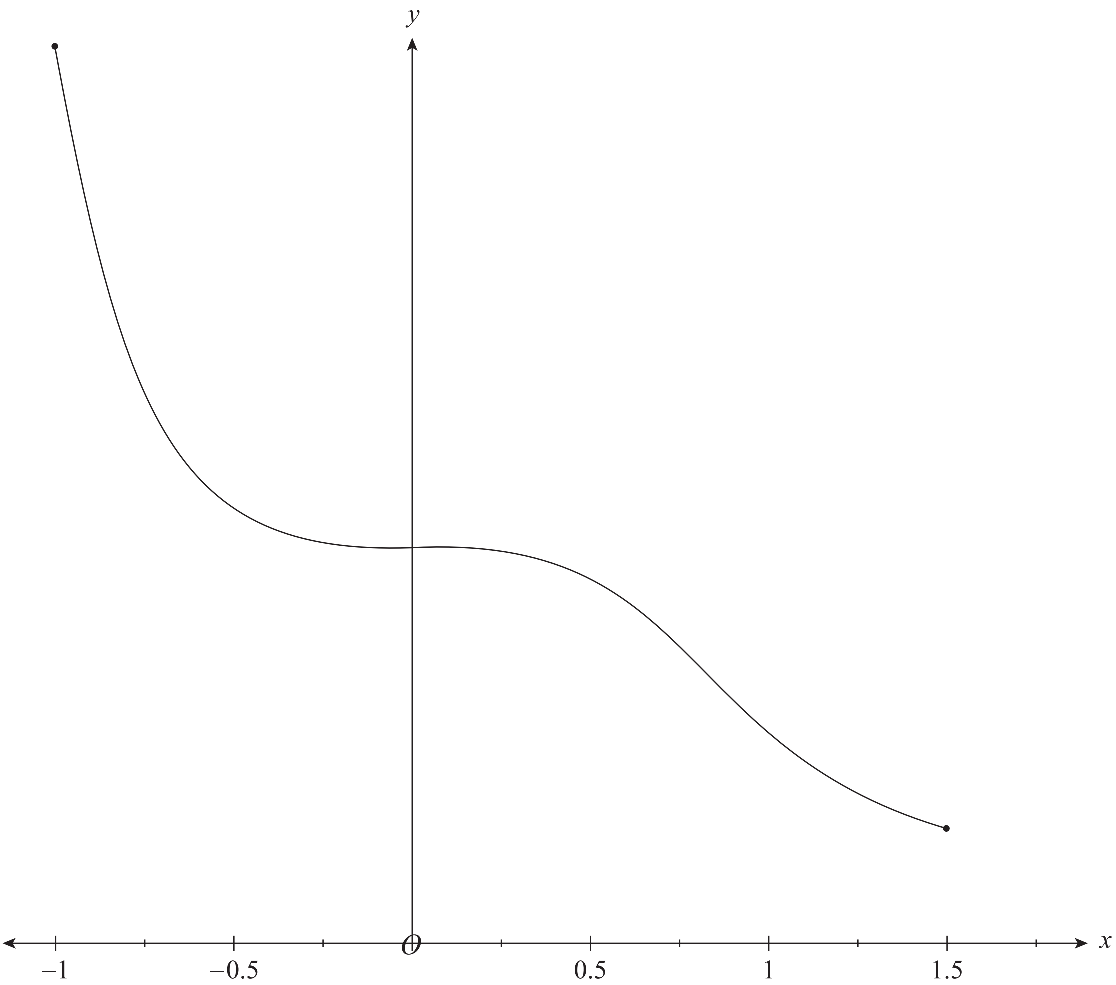
(1 mark)

PLEASE TURN OVER

QUESTION 7

Consider the graph of $y = f(x)$, as drawn below for $-1 \le x \le 1.5$.

This graph has one stationary point, at the point where $x = 0$, and two inflection points, at the points where $x = 0$ and $x = 0.874$.



(a) Tick the appropriate box to indicate which one of the following statements is true for $f'(x)$ for all x values $-1 \le x \le 1.5$.

$f'(x) > 0$. ☐

$f'(x) \ge 0$. ☐

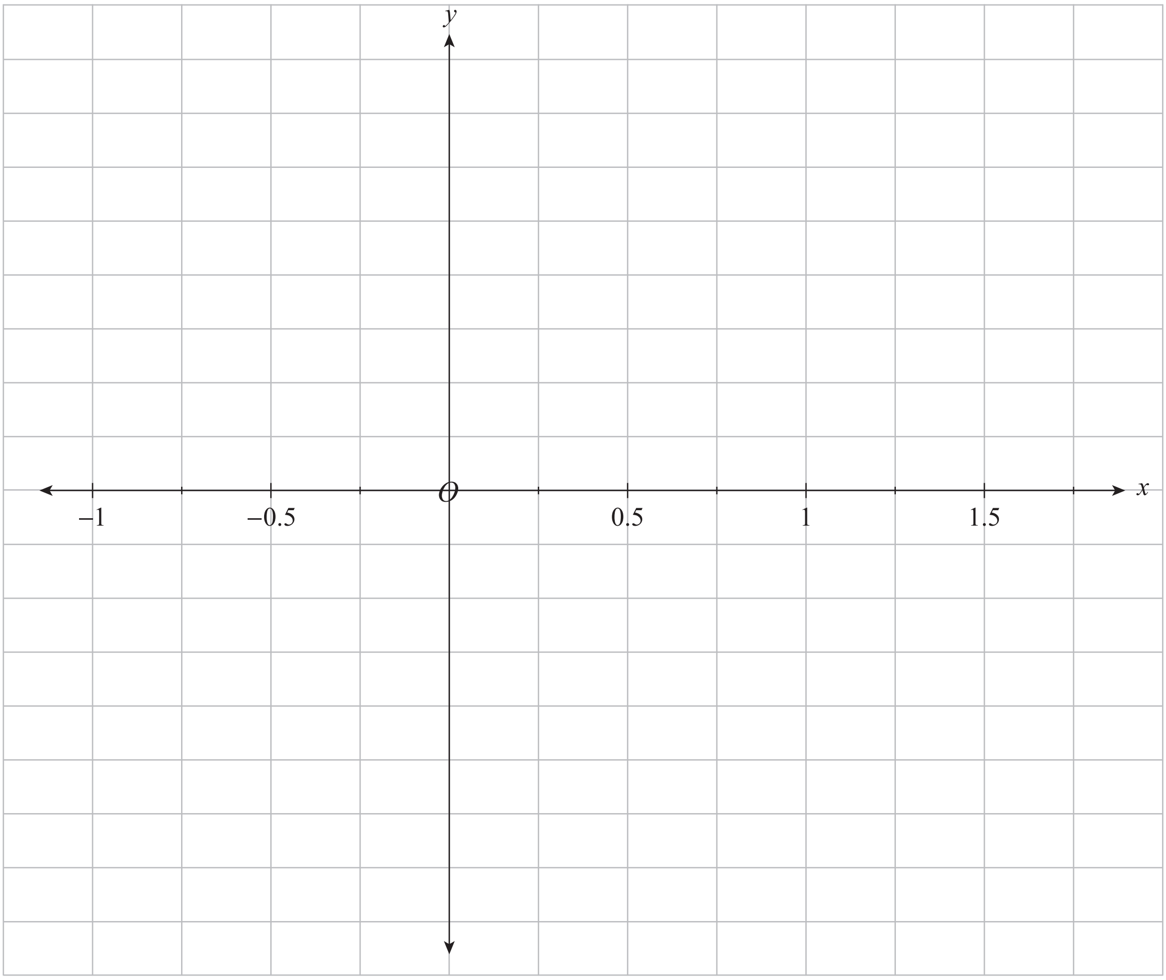
$f'(x) < 0$. ☐

$f'(x) \le 0$. ☐

$f'(x)$ takes positive, negative, and zero values. ☐

(1 mark)

(b) On the axes below, sketch the graph of $y = f'(x)$ for $-1 \leq x \leq 1.5$.



(3 marks)

(c) Tick the appropriate box to indicate which one of the following statements is true for $f''(x)$ for all x values $-1 \leq x \leq 1.5$.

$f''(x) > 0$. ☐

$f''(x) \geq 0$. ☐

$f''(x) < 0$. ☐

$f''(x) \leq 0$. ☐

$f''(x)$ takes positive, negative, and zero values. ☐

(1 mark)

QUESTION 8

A consumer group is interested in a comparison of two different brands of AA battery.

For one brand of AA battery (alpha batteries) the cost of a single battery is $0.99.

The *running time*, in hours, of a single alpha battery operating a specific electronic device has a mean of $\mu_A = 8.1$ hours and a standard deviation of $\sigma_A = 0.25$ hours.

For a cheaper brand of AA battery (budget batteries) the cost of a single battery is $0.90.

A test is conducted to compare the running times of budget batteries and alpha batteries. A sample of ten budget batteries is randomly selected and the length of time for which each battery operates the same specific electronic device is recorded.

(a) The running times, in hours, of the ten budget batteries are shown below:

8.2	7.4	8.5	7.6	8.1
7.8	8.3	8.2	7.9	7.8

(i) Find the sample mean for these ten running times.

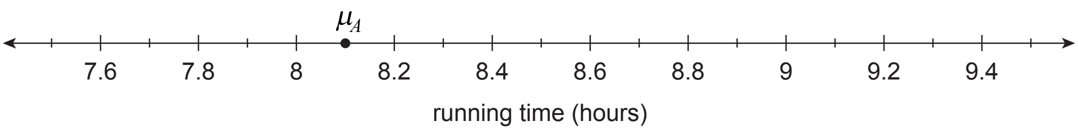
(1 mark)

The running time of budget batteries is normally distributed, with a mean of μ_B and a standard deviation of $\sigma_B = 0.25$ hours.

(ii) Construct a 95% confidence interval for μ_B.

(2 marks)

(b) The mean running time μ_A of alpha batteries is shown on the number line below:



(i) On the number line above, draw the confidence interval for μ_B that you found in part (a)(ii), showing the sample mean and the lower and upper boundaries of the confidence interval.

(1 mark)

(ii) What can you conclude from your confidence interval about the mean running time of budget batteries compared with alpha batteries? Give a reason for your answer.

(2 marks)

To take into account the different costs of the two brands of battery, their running time per dollar (their 'dollar performance') is calculated. For example, a battery that cost \$0.90 and had a running time of 8.2 hours would have a dollar performance of

$$\frac{8.2 \text{ hours}}{\$0.90} = 9.11 \text{ hours / dollar.}$$

The table below shows the running times and dollar performance of the same sample of ten budget batteries:

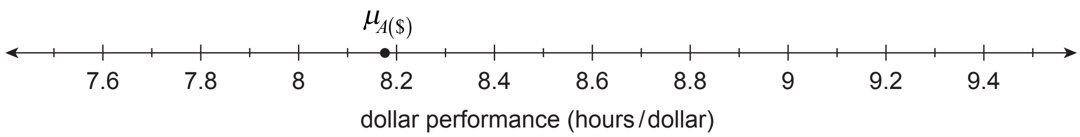
Battery number	Running time (hours)	Dollar performance (hours / dollar)
1	8.2	9.11
2	7.4	8.22
3	8.5	9.44
4	7.6	8.44
5	8.1	9.00
6	7.8	8.67
7	8.3	9.22
8	8.2	9.11
9	7.9	8.78
10	7.8	8.67

The dollar performance of budget batteries is normally distributed, with a standard deviation of $\sigma_{B(\$)} = 0.278$ hours.

(c) Construct a 95% confidence interval for the mean dollar performance of budget batteries.

(2 marks)

(d) The mean dollar performance of alpha batteries is $\mu_{A(\$)} = \dfrac{\text{8.1 hours}}{\$0.99} = 8.18$ hours / dollar, as shown on the number line below:



(i) On the number line above, draw the confidence interval that you found in part (c), showing the sample mean and the lower and upper boundaries of the confidence interval.

(1 mark)

(ii) What can you conclude from your confidence interval about the mean dollar performance of budget batteries compared with alpha batteries? Give a reason for your answer.

(2 marks)

QUESTION 9

(a) Show that $\sqrt{p}-\sqrt{q}=\dfrac{p-q}{\sqrt{p}+\sqrt{q}}$ for positive real numbers p and q.

(2 marks)

(b) Hence or otherwise, find, from *first principles*, $f'(x)$ if $f(x)=\sqrt{3-x}$.

(4 marks)

PLEASE TURN OVER

QUESTION 10

The calculation of IQ (intelligence quotient) is a way of comparing the intelligence of individuals with the intelligence of the general population. As part of the process of comparison, social and economic factors are taken into account. As a result, the mean IQ of the general population is 100 points for the purposes of comparison.

A study of cognitive development measured the IQ of a random sample of 1300 children who had been breastfed for 6 months or more.

The sample mean for the IQ of these children was $\bar{x} = 103.16$ points.

(a) On the basis of this study, undertake a two-tailed Z-test at the 0.05 level of significance to test the null hypothesis that, on average, breastfed children have an IQ that is no different from that of the general population. Assume that IQ has a standard deviation of $\sigma = 15$.

(4 marks)

(b) With reference to your result in part (a), and to the value of $\bar{x}$, what can you conclude about the IQ of children who have been breastfed for 6 months or more, relative to the general population?

(2 marks)

(c) A child who has been breastfed for more than 6 months is chosen at random.

(i) Tick the appropriate box to indicate which one of the following statements is compatible with your results in parts (a) and (b).

This child will have an IQ of more than 100 points. ☐

This child will have an IQ of less than 100 points. ☐

This child will have an IQ of 100 points. ☐

It cannot be concluded whether this child will have an IQ of more or less than 100 points. ☐

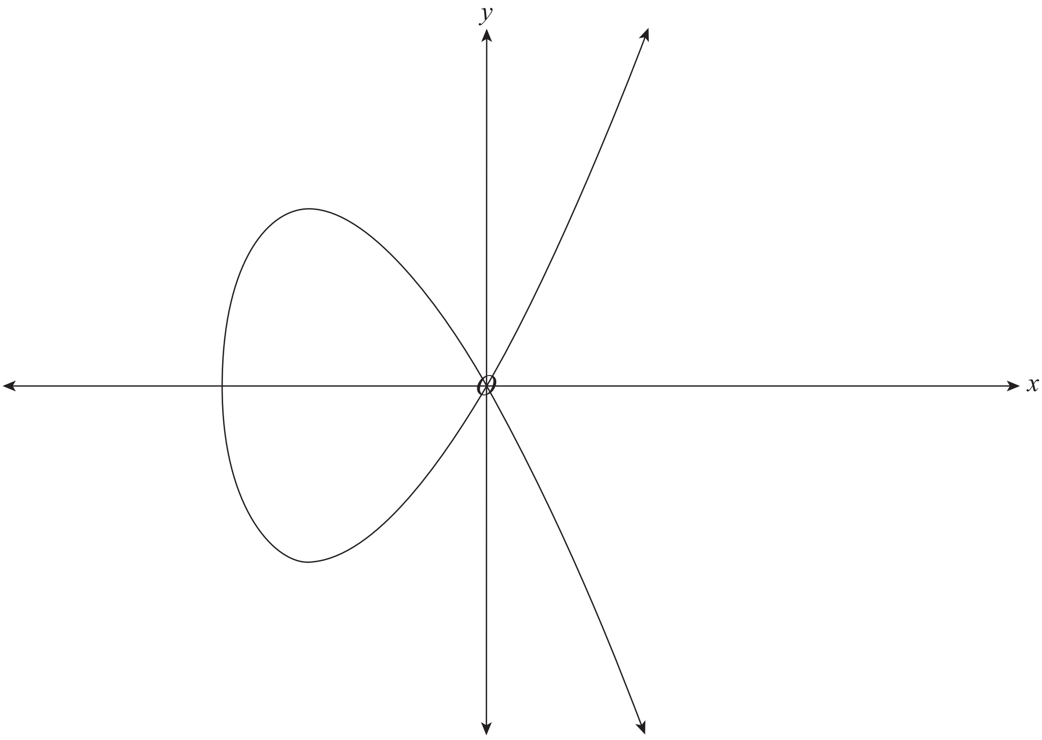
(1 mark)

(ii) Give a reason for your answer to part (c)(i).

(1 mark)

QUESTION 11

The implicit relation represented by the equation $x^3 = y^2 - 3x^2$ is graphed below:



(a) Show that $\frac{dy}{dx} = \frac{3x^2 + 6x}{2y}$.

(2 marks)

(b) (i) The graph of the relation $x^3 = y^2 - 3x^2$ has two horizontal tangents to the left of the origin.

Find the equation of the horizontal tangent that lies **below** the x-axis.

(4 marks)

(ii) Find the x-coordinate of the point where this horizontal tangent intersects the curve again.

(2 marks)

(c) The graph of this relation has a single vertical tangent. Find its equation.

(3 marks)

QUESTION 12

Between 2008 and 2010 a number of animals were used to predict the result of international football games.

Paul the Octopus was the best known of these animals. To make the prediction, Paul was presented with two parcels of food, each marked with the colours of one of the two competing teams. The parcel of food that Paul ate first was taken as his prediction of the winning team.

Source: © ROLAND WEIHRAUCH/epa/Corbis

These were 'elimination' football games, resulting in a win for one of the two teams.

Assume that Paul selected his parcel of food at random.

(a) What is the probability that Paul made a correct selection in a single such game?

(1 mark)

(b) This method of prediction was repeated for the result of fourteen elimination football games.

What is the probability that Paul made twelve or more correct selections in these fourteen games?

(2 marks)

(c) Including Paul the Octopus, fifty animals were used to predict the result of these fourteen elimination football games, using the same method of prediction.

What is the probability that at least one of the animals made twelve or more correct selections in these fourteen games?

(3 marks)

(d) Paul the Octopus attracted media attention by making twelve correct selections in fourteen elimination football games. One newspaper report suggested that Paul's 'predictions' were too accurate to be explained by random selection, and that he must therefore be psychic.

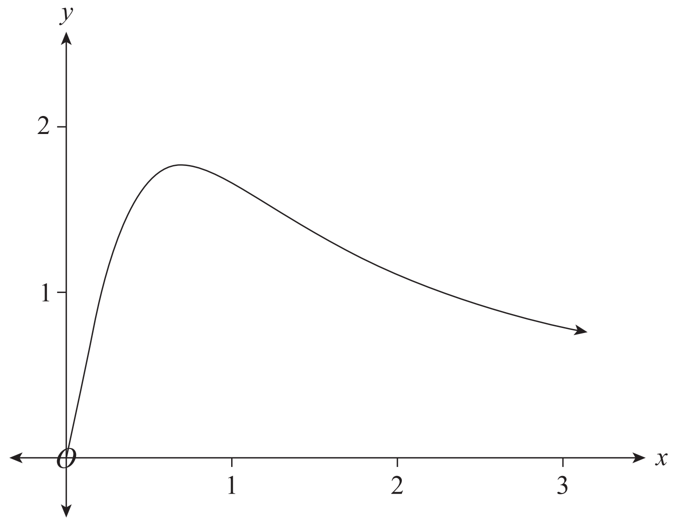
Comment on the validity of the newspaper's suggestion with reference to your answers to parts (b) and (c).

(2 marks)

PLEASE TURN OVER

QUESTION 13

For $f(x)=\dfrac{5x}{2x^2+1}$ with $x \geq 0$, the graph of $y=f(x)$ is shown below:



(a) On the diagram above, shade the region that corresponds to $\displaystyle\int_0^2 \frac{5x}{2x^2+1}\,dx$.

(1 mark)

(b) (i) Show *by integration* that $\displaystyle\int \frac{5x}{2x^2+1}\,dx = \frac{5}{4}\ln\left(2x^2+1\right)+c$.

(2 marks)

(ii) Hence find the exact value of $\displaystyle\int_0^2 \frac{5x}{2x^2+1}\,dx$.

(1 mark)

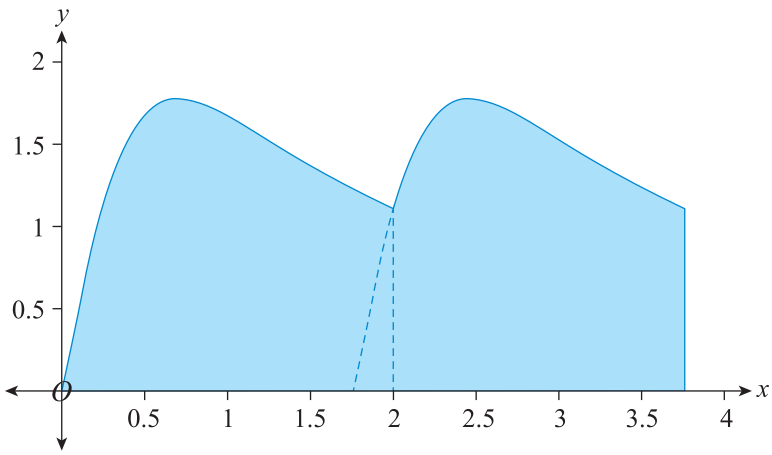
(c) (i) Evaluate $f(2)$.

(1 mark)

(ii) Find another value of x such that $f(x) = f(2)$.

(2 marks)

(d) An image is constructed by shading two overlapping copies of the region referred to in part (a), as shown below:



(i) Write down an integral expression that corresponds to the area of this image.

(2 marks)

PLEASE TURN OVER

(ii) Hence find an exact value for the area of the image in the form $A \ln B$, where A and B are rational numbers.

(3 marks)

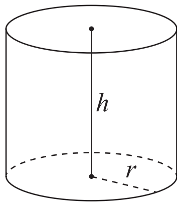
QUESTION 14

Ben is a hydrogeologist who has drilled a cylindrical bore with a radius of 8 centimetres to a depth of 35 metres.

Cylinder

Volume (V)

$V = \pi r^2 h$



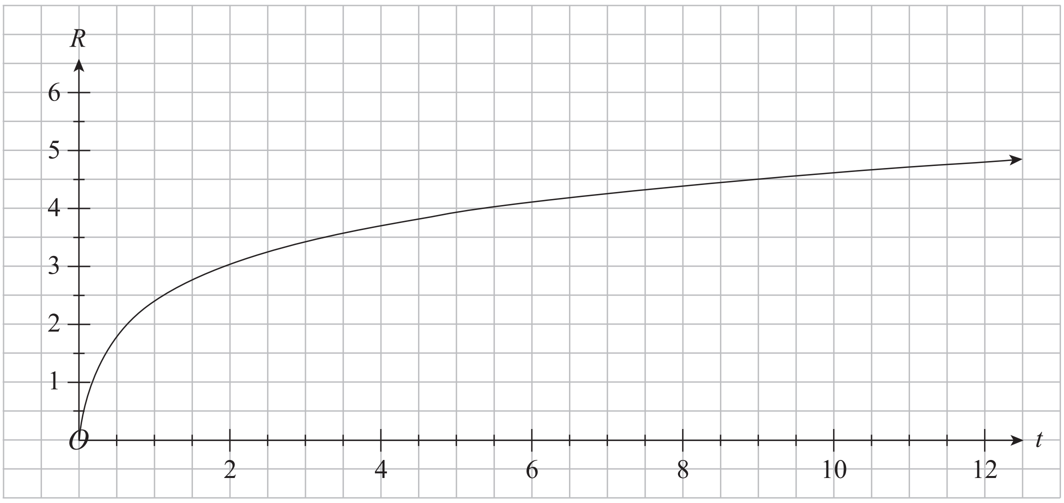
(a) If 1 litre of water has a volume of 1000 cubic centimetres, find the volume of the bore in litres.

(2 marks)

Underground water flows into the bore. Ben pumps out all the water from the bore so that at $t = 0$ the bore contains no water. He then turns off the pump and the bore starts to fill with water. The net rate of water flow, $R(t)$, into the bore can be modelled by the logarithmic function

$$R(t) = \ln(10t + 1) \text{ litres / minute.}$$

A graph of this rate function is shown below:



PLEASE TURN OVER

Ben is interested in the volume of water in the bore t minutes after he turns off the pump.

(b) (i) How many litres of water are in the bore 10 minutes after the pump is turned off?

(2 marks)

(ii) Complete the following table.

Time after the pump is turned off (min)	Volume of water in the bore (L)
10	
50	
80	456
100	

(2 marks)

(c) The volume, in litres, of water in the bore t minutes after the pump is turned off can be modelled by the function $V(t)$.

Write an equation for the relationship between $V(t)$ and $R(t)$.

(1 mark)

(d) Consider $V(t) = t\ln(10t+1) - t + \frac{1}{10}\ln(10t+1)$.

Verify by differentiation that $V(t)$ satisfies the equation that you wrote in part (c).

(4 marks)

(e) Find the time taken until the bore contains 650 litres of water.

(2 marks)

QUESTION 15

Let $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$.

(a) (i) Find $f'(x)$.

(1 mark)

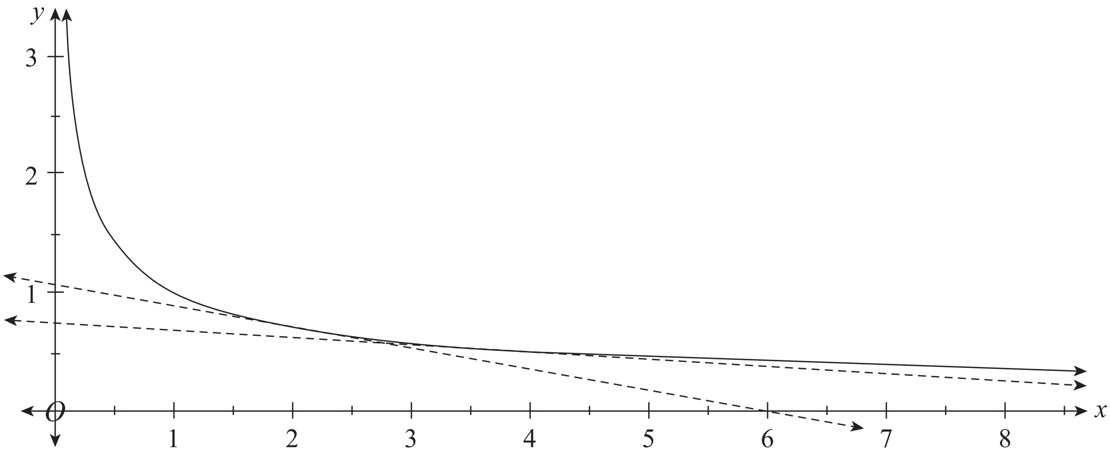
(ii) Find the equation of the tangent to the graph of $y = f(x)$ at $x = 1$.

(2 marks)

(iii) Find the x-intercept of the tangent to the graph of $y = f(x)$ at $x = 1$.

(1 mark)

The graph of $y = f(x)$ is shown below. The tangents to the graph of $y = f(x)$ at $x = 2$ and $x = 4$ are also shown.



(b) (i) The tangent to the graph of $y = f(x)$ at $x = 4$ has equation

$$y = -\frac{1}{16}x + \frac{3}{4}.$$

Find the x-intercept of this tangent.

(1 mark)

(ii) The tangent to the graph of $y = f(x)$ at $x = 6$ has equation

$$y = -\frac{1}{12\sqrt{6}}x + \frac{3}{2\sqrt{6}}.$$

Find the x-intercept of this tangent.

(1 mark)

(c) Consider the tangent to the graph of $y = f(x)$ at $x = a$.

Let x_a be the x-intercept of this tangent.

(i) On the basis of your answers to parts (a) and (b), make a conjecture describing how x_a is related to a.

(1 mark)

PLEASE TURN OVER

(ii) Prove or disprove the conjecture that you made in part (c)(i).

(5 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 1(a)(ii) continued').

Question 16 begins on page 34.

QUESTION 16

A scientist is studying an isolated population of a species of native Australian marsupial.

In terms of breeding, there are three groups of interest in this population of marsupials: female pups, immature females (1 year old), and mature females (2 years old or older).

On the basis of her research, the scientist makes the following assumptions about these groups:

- After a year, half of the female pups have survived and become immature females.
- After a year, one-third of the immature females have survived and become mature females.
- Each year mature females breed and produce four female pups.
- 9% of mature females survive and breed in the following year.
- Female pups and immature females do not produce any female pups.

Source: © iStockphoto.com/alantobey

Using the information above, the scientist constructs the following matrices. Matrix P represents the females in this population of marsupials in 2009:

$$L = \begin{bmatrix} 0 & 0 & 4 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & \frac{9}{100} \end{bmatrix} \text{ and } P = \begin{bmatrix} 130 \\ 60 \\ 20 \end{bmatrix}.$$

(a) (i) Evaluate LP.

(1 mark)

(ii) Explain what you have calculated in part (a)(i) above.

(1 mark)

(b) What will happen to this population of marsupials in the long term? Provide evidence for your answer.

(2 marks)

The scientist finds that a second species of native Australian marsupial survives and breeds in a slightly different way.

The number of females in a population of this species after 1 year can be found by using the matrix product MX, where:

$$M = \begin{bmatrix} 0 & k & 3 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{bmatrix}.$$

(c) List the ways in which the survival and breeding of the second species of marsupial differ from those of the species of marsupial described on page 34.

(2 marks)

(d) (i) Now consider the matrix $X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

If matrix X represents the numbers of female pups, immature females, and mature females in a particular year, comment on the significance of the matrix equation $MX = X$.

(1 mark)

(ii) By considering the matrix equation $MX = X$, obtain the augmented matrix

$$\left[\begin{array}{ccc:c} 1 & -k & -3 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right].$$

(3 marks)

(iii) By applying clearly defined row operations to the augmented matrix in part (d)(ii) opposite, find the value of k for which this system of equations has infinitely many solutions.

(3 marks)

(iv) For this value of k, give the solution to the system of equations in parametric form.

(2 marks)

(v) In a remote population of the second species of marsupial there are 300 females in total. This population is in a state of equilibrium.

How many mature females are there in the population?

(2 marks)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 1(a)(ii) continued').

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 1(a)(ii) continued').

You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 MATHEMATICAL STUDIES

Standardised Normal Distribution

A measurement scale X is transformed into a standard scale Z, using the formula

$$Z = \frac{X-\mu}{\sigma}$$

where μ is the population mean and σ is the standard deviation for the population distribution.

Confidence Interval — Mean

A 95% confidence interval for the mean μ of a normal population with standard deviation σ, based on a simple random sample of size n with sample mean $\bar{x}$, is

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}.$$

For suitably large samples, an approximate 95% confidence interval can be obtained by using the sample standard deviation s in place of σ.

Sample Size — Mean

The sample size n required to obtain a 95% confidence interval of width w for the mean of a normal population with standard deviation σ is

$$n = \left(\frac{2\times 1.96\sigma}{w}\right)^2.$$

Confidence Interval — Population Proportion

An approximate 95% confidence interval for the population proportion p, based on a large simple random sample of size n with sample proportion

$\hat{p} = \frac{X}{n}$, is

$$\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Sample Size — Proportion

The sample size n required to obtain an approximate 95% confidence interval of approximate width w for a proportion is

$$n = \left(\frac{2\times 1.96}{w}\right)^2 p^*\left(1-p^*\right).$$

(p^* is a given preliminary value for the proportion.)

Binomial Probability

$$P(X=k) = C_k^n p^k (1-p)^{n-k}$$

where p is the probability of a success in one trial and the possible values of X are $k = 0, 1, \ldots n$ and

$$C_k^n = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\ldots(n-k+1)}{k!}.$$

Binomial Mean and Standard Deviation

The mean and standard deviation of a binomial count X and a proportion of successes $\hat{p} = \frac{X}{n}$ are

$$\mu_X = np \qquad \mu(\hat{p}) = p$$

$$\sigma_X = \sqrt{np(1-p)} \qquad \sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

where p is the probability of a success in one trial.

Matrices and Determinants

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = |A| = ad - bc$ and

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Derivatives

$f(x) = y$	$f'(x) = \frac{dy}{dx}$
x^n	nx^{n-1}
e^{kx}	ke^{kx}
$\ln x = \log_e x$	$\frac{1}{x}$

Properties of Derivatives

$$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Quadratic Equations

If $ax^2 + bx + c = 0$ then $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$