Government
of South Australia

SACE
Board of SA

External Examination 2012

2012 MATHEMATICAL METHODS

FOR OFFICE USE ONLY

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ATTACH SACE REGISTRATION NUMBER LABEL
TO THIS BOX

Graphics calculator

Brand

Model

Computer software

Friday 9 November: 9 a.m.

Time: 3 hours

Pages: 33
Questions: 13

Examination material: one 33-page question booklet
one SACE registration number label

Approved dictionaries, notes, calculators, and computer software may be used.

Instructions to Students

1. You will have 10 minutes to read the paper. You must not write in your question booklet or use a calculator during this reading time but you may make notes on the scribbling paper provided.

2. Answer ***all*** parts of Questions 1 to 13 in the spaces provided in this question booklet. There is no need to fill all the space provided. You may write on pages 8, 15, and 32 if you need more space, making sure to label each answer clearly.

3. The total mark is 153. The allocation of marks is shown below:

Question	1	2	3	4	5	6	7	8	9	10	11	12	13
Marks	9	6	11	8	11	18	13	14	11	9	14	17	12

4. Appropriate steps of logic and correct answers are required for full marks.

5. Show all working in this booklet. (You are strongly advised ***not*** to use scribbling paper. Work that you consider incorrect should be crossed out with a single line.)

6. Use only black or blue pens for all work other than graphs and diagrams, for which you may use a sharp dark pencil.

7. State all answers correct to three significant figures, unless otherwise stated or as appropriate.

8. Diagrams, where given, are not necessarily drawn to scale.

9. The list of mathematical formulae is on page 33. You may remove the page from this booklet before the examination begins.

10. Complete the box on the top right-hand side of this page with information about the electronic technology you are using in this examination.

11. Attach your SACE registration number label to the box at the top of this page.

QUESTION 1

Find $\frac{dy}{dx}$ without simplifying when:

(a) $y = x^3 - 3x^2 + 4$.

(2 marks)

(b) $y = \ln(2x - 1)$.

(2 marks)

(c) $y = (3x^2 + 1)^5$.

(2 marks)

(d) $y = (4 + x)e^{2x}$.

(3 marks)

QUESTION 2

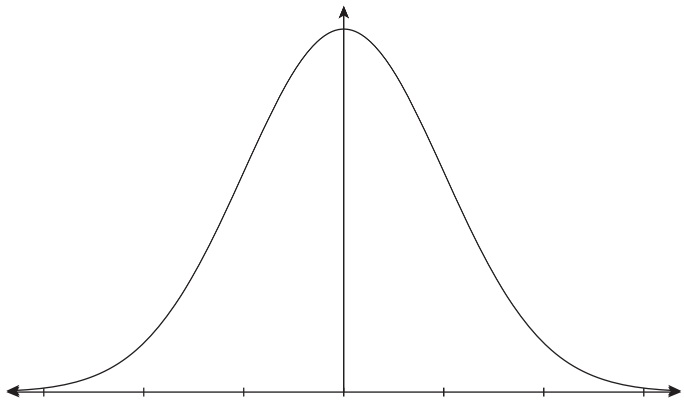
The Australorp chicken produces eggs with weights (in grams) that are normally distributed with a mean $\mu = 56$ grams and a standard deviation $\sigma = 4$ grams. The eggs are classified as medium, large, and extra large according to their weight as shown in the table below.

Class	*Weight (grams)*
Medium	$W < 54$
Large	$54 \leq W < 60$
Extra large	$W \geq 60$

This photograph of "Grade A" Egg Scale cannot be reproduced here for copyright reasons.

Source: © Nasco

(a) The normal distribution of the weights of Australorp chicken eggs is graphed below. Clearly write on the horizontal axis the numerical values of $\mu, \mu \pm \sigma, \mu \pm 2\sigma$, and $\mu \pm 3\sigma$.



(2 marks)

(b) On the graph in part (a), clearly shade the area that represents the proportion of Australorp chicken eggs that are extra large.

(1 mark)

(c) An egg is chosen at random; find the probability that the egg is:

(i) medium.

(1 mark)

(ii) large.

(1 mark)

(d) There is a probability of 0.2 that an egg weighs more than k grams. Find k.

(1 mark)

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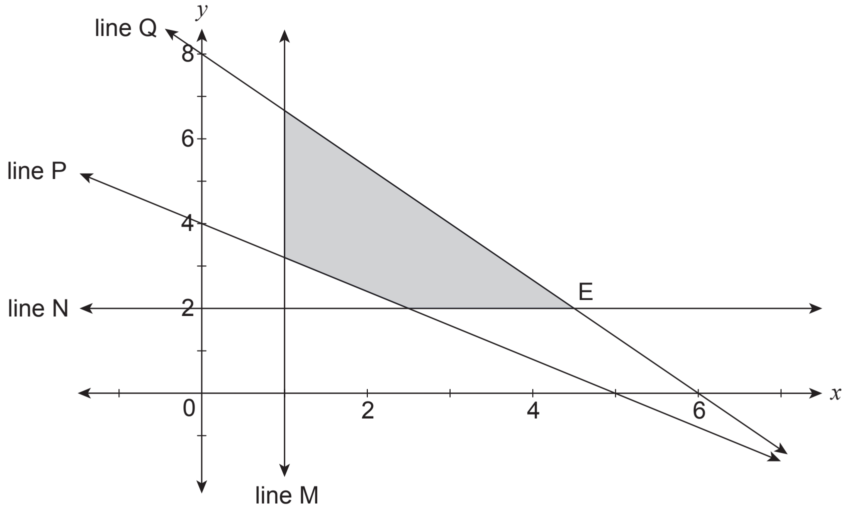
QUESTION 3

Consider the following graphs:



(a) Determine the equation of:

(i) line M.

(1 mark)

(ii) line N.

(1 mark)

(iii) line P in the form $Ax + By = C$, given that the slope of this line is $\frac{-4}{5}$.

(2 marks)

(iv) line Q in the form $Ax + By = C$.

(3 marks)

(b) Write four inequalities to define the feasible region shaded in the graph opposite.

(2 marks)

(c) If the point E $\left(\frac{9}{2}, 2\right)$ is the optimal solution for the objective function $p = 3x + y$, find the value of k such that $p = 3x + ky$ has more than one optimal solution, including E.

(2 marks)

PLEASE TURN OVER

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 3(a)(i) continued').

QUESTION 4

Let $A=\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} x & 2 \\ y & -1 \end{bmatrix}$.

(a) Evaluate:

(i) $3A$.

(1 mark)

(ii) A^3.

(2 marks)

(iii) AB.

(2 marks)

(b) If $AB=\begin{bmatrix} 7 & 3 \\ -2 & -4 \end{bmatrix}$, find the values of x and y.

(3 marks)

QUESTION 5

An unbiased octahedral die with the numbers 1 to 8 on the faces is rolled.

(a) The die is rolled once. What is the probability of the 7 being in the position shown?

(1 mark)

(b) The die is rolled twice. What is the probability of the 7 being in the position shown both times?

(1 mark)

(c) The die is rolled twenty times.

(i) Calculate $C_3^{20}(0.125)^3(0.875)^{17}$.

(2 marks)

(ii) Explain the meaning of $C_3^{20}(0.125)^3(0.875)^{17}$ with respect to the die being rolled twenty times.

(2 marks)

(d) Determine the probability that, in twenty rolls of the octahedral die:

(i) the 7 will be in the position shown fewer than five times.

(2 marks)

(ii) the 7 will be in the position shown at least twice and no more than six times.

(3 marks)

QUESTION 6

A farmer realised that a locust plague was approaching. The number of locusts (in thousands) infecting a crop n days after the crop was sprayed with a slow-acting yet powerful insecticide is modelled by the formula:

$$P(n) = -0.07n^3 + 0.63n^2 + 3.36n + 0.042$$

where $0 \leq n \leq 13$.

(a) (i) How many locusts were present when the crop was sprayed?

(1 mark)

(ii) How many locusts were present 6 days after the crop was sprayed?

(2 marks)

(b) (i) Solve for n if $P(n) = 0$.

(1 mark)

(ii) Interpret your answer to part (b)(i).

(2 marks)

(c) (i) Determine an expression for $P'(n)$.

(2 marks)

(ii) Find $P'(3)$.

(1 mark)

(iii) Interpret your answer to part (c)(ii).

(2 marks)

(d) (i) Solve for n if $P'(n)=0$.

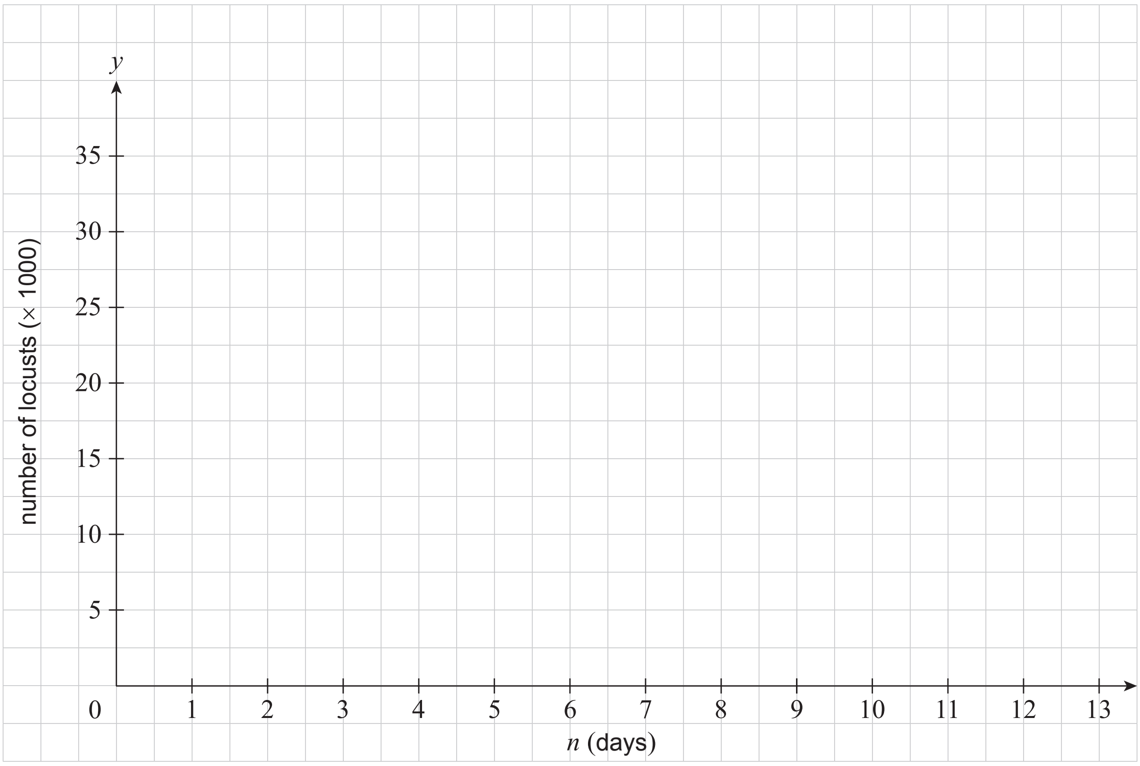
(1 mark)

(ii) Interpret your answer to part (d)(i).

(2 marks)

(e) On the set of axes below, draw the graph of $y = P(n)$ for $0 \le n \le 13$, where

$$P(n) = -0.07n^3 + 0.63n^2 + 3.36n + 0.042.$$



(2 marks)

(f) On the graph in part (e) clearly label:

(i) the point A, where $P'(n) = 0$.

(1 mark)

(ii) the point B, where the number of locusts is 0.

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 3(a)(i) continued').

QUESTION 7

The weights of nectarines produced in a particular orchard are normally distributed with a mean of 78 grams and a standard deviation of 0.9 grams.

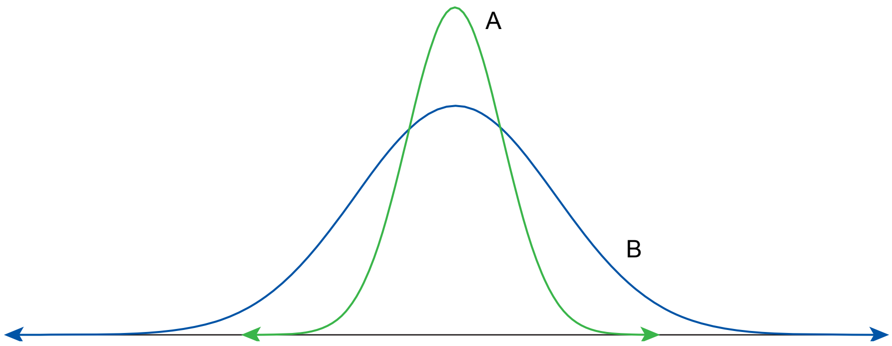
They are sold to fruiterers in trays that contain forty-eight nectarines that are either ripe or under-ripe.

Let $\overline{W}_{48}$ be the distribution of the average weight of nectarines in a randomly chosen tray.

The two curves A and B below show the distribution of the weights of individual nectarines, W, and the distribution of $\overline{W}_{48}$.

This photograph of nectarines cannot be reproduced here for copyright reasons

Source: © Unlimited Group



(a) Which curve represents the distribution of $\overline{W}_{48}$? Explain.

(2 marks)

(b) Determine the mean and standard deviation of the average weight of nectarines on each tray.

(2 marks)

(c) Find the probability that a randomly chosen tray will have a total weight exceeding 3750 grams.

(2 marks)

(d) It is known that 12% of all nectarines on the trays are under-ripe.

(i) Explain why the distribution of the number of under-ripe nectarines on a tray is binomial.

(2 marks)

(ii) Find the mean and standard deviation of this distribution.

(2 marks)

(iii) Use a normal approximation to find the probability that a tray has at least ten nectarines that are under-ripe.

(3 marks)

QUESTION 8

Enter the following data into your graphics calculator.

t (minutes)	0	1	2	3	4	5	6	7	8	9
T (°C)	80	72	64	58	52	47	43	40	36	33

(a) (i) In terms of T and t, determine the equation of the line of best fit in the form $T = mt + C$.

(2 marks)

(ii) Find the r value.

(1 mark)

(b) Describe the linear relationship between T and t.

(2 marks)

(c) (i) Estimate the value of T when $t = 10$.

(1 mark)

(ii) Comment on the reliability of your answer to part (c)(i).

(2 marks)

The values in the table represent the temperature of a cup of soup as it cools, so it is likely that an exponential equation will provide a better model of the relationship between T and t than a linear equation.

(d) Suppose $T = 78.4e^{-0.098t}$.

(i) Determine an expression for $\frac{dT}{dt}$.

(2 marks)

(ii) Find the value of $\frac{dT}{dt}$ when $t = 4$.

(1 mark)

(iii) Describe what is happening to the temperature of the soup when $t = 4$.

(3 marks)

QUESTION 9

As part of a fundraiser for a local community centre, a group decided to make wheat bags to sell. To keep the construction and design simple, the group decided to make only two types of bags: a square bag (18 cm × 18 cm) and a rectangular bag (18 cm × 35 cm). The square bag uses 300 grams of wheat and the rectangular bag uses 600 grams of wheat.

Let x be the number of square bags made and y be the number of rectangular bags made.

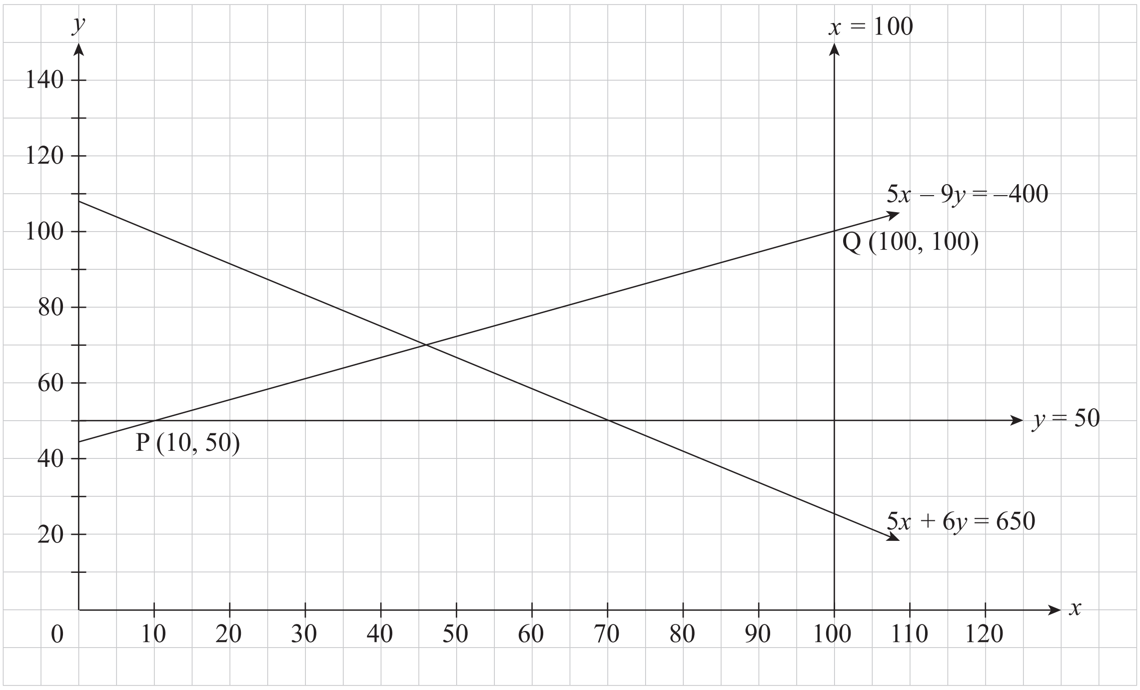
The feasible region for the production of wheat bags is defined by the following constraints:

$$x \leq 100$$
$$y \geq 50$$
$$5x + 6y \geq 650$$
$$5x - 9y \geq -400.$$

(a) Explain the meaning of $x \leq 100$.

(1 mark)

(b) (i) The limits of the constraints are illustrated on the following graph. Shade in the feasible region.



(2 marks)

(ii) State the coordinates of the four vertices of the feasible region.

(3 marks)

(c) It will cost $9 to produce a square wheat bag and $12 to produce a rectangular wheat bag.

(i) State the objective cost function.

(1 mark)

(ii) Calculate the optimal solution for minimising the cost.

(2 marks)

(iii) Determine the quantity of wheat (in kg) that will be required to make the optimal number of each wheat bag to minimise the cost.

(2 marks)

QUESTION 10

A council survey of 200 local residents found that 162 of them support placing speed humps on a road known for drag racing.

(a) What percentage of these residents support placing speed humps on this road?

(1 mark)

(b) Construct a 95% confidence interval for the true proportion of all local residents who support placing speed humps on this road.

(2 marks)

(c) A council press release stated that 'more than 80% of local residents support placing speed humps on this road'. Comment on this statement.

(2 marks)

(d) Determine the smallest sample size necessary to ensure the width of the 95% confidence interval for the percentage of local residents who support placing speed humps on this road is no greater than 6%.

(3 marks)

(e) What happens to the width of the 95% confidence interval if the sample size is increased?

(1 mark)

QUESTION 11

The share price (in dollars) for the Aasiya Oil company over a period of 20 months can be represented as:

$$A(t) = 0 \cdot 45(1 \cdot 08)^t$$

where t is the number of months since January 2010.

(a) (i) Determine the share price for Aasiya Oil, to the nearest cent, when $t = 6$.

(1 mark)

(ii) How many months will it take for the share price of Aasiya Oil to exceed $1.00?

(2 marks)

The share price (in dollars) for the Blaed Oil company over a period of 20 months can be represented as:

$$B(t) = 5 \cdot 8(0 \cdot 92)^t$$

where t is the number of months since January 2010.

(b) (i) State the initial share price for Blaed Oil.

(1 mark)

(ii) Interpret $(0 \cdot 92)^t$.

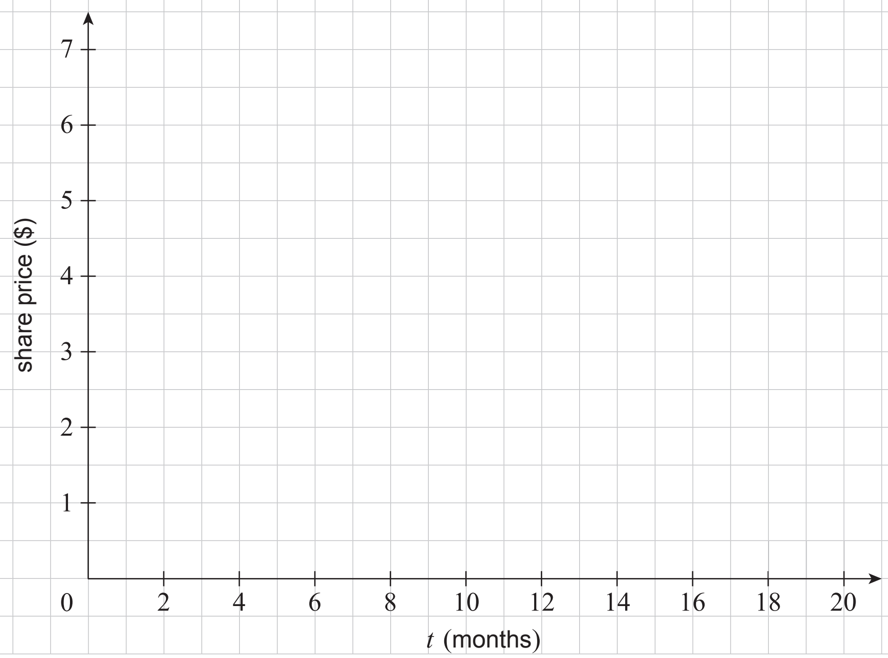
(2 marks)

(c) (i) Using the laws of logarithms, clearly show that $t = 15.94$ when:

$$5 \cdot 8(0 \cdot 92)^t = 0 \cdot 45(1 \cdot 08)^t.$$

(3 marks)

(ii) Draw the graphs of $A(t)$ and $B(t)$ using the set of axes below and clearly label the point of intersection, P.



(3 marks)

(iii) Interpret the point of intersection identified in part (c)(ii).

(2 marks)

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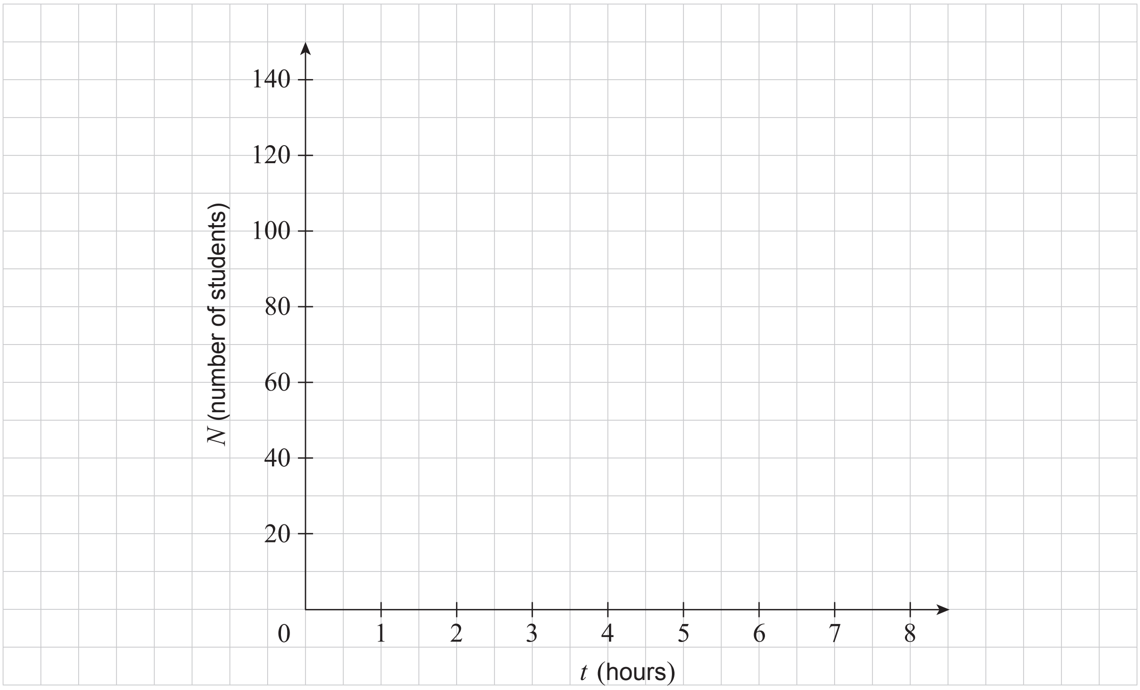
QUESTION 12

An after-party for a Year 12 formal event was planned by a group of eight students. The number of students N who had heard about the party after t hours is shown in the following table.

Time (t hours)	0	0.5	1	2	3	4	6	7	8
Number of students (N)	8	13	22	50	87	116	137	138	139

(a) (i) Plot the values displayed in the table above on the set of axes below.



(2 marks)

(ii) What model best describes the graph in part (a)(i)?

(1 mark)

(iii) Use the table of values and the graph in part (a)(i) to predict the number of Year 12 students at the school.

(1 mark)

(b) (i) Rule a straight line through the points $(1, 22)$ and $(3, 87)$ on the grid in part (a)(i).

(1 mark)

(ii) Determine the slope of this line.

(1 mark)

(iii) Interpret your answer to part (b)(ii).

(2 marks)

(c) The model that best describes the data in the table was found to be:

$$N(t) = \frac{140}{1 + 16.5e^{-1.1t}}.$$

(i) Find $N(0)$.

(1 mark)

(ii) Find $N(12)$.

(1 mark)

(iii) Interpret your answer to part (c)(ii).

(1 mark)

(d) (i) Given the model:

$$N(t) = \frac{140}{1 + 16.5e^{-1.1t}}$$

find an expression for $N'(t)$.

(3 marks)

(ii) Find $N'(2.4)$.

(1 mark)

(iii) Interpret your answer to part (d)(ii).

(2 marks)

QUESTION 13

A company that produces and markets uniforms has three manufacturing outlets: one in New South Wales, one in Western Australia, and one in South Australia. Among other items, each outlet manufactures shirts, blazers, and jumpers. Workers are needed to cut fabric, sew the pieces together, and finish the article of clothing.

Matrix A shows the time (in hours) of each type of labour required to manufacture a shirt, a blazer, and a jumper.

$$A = \begin{array}{l} \text{shirt} \\ \text{blazer} \\ \text{jumper} \end{array} \overset{\begin{array}{ccc}\text{cutting} & \text{sewing} & \text{finishing}\end{array}}{\begin{bmatrix} 0.25 & 0.40 & 0.50 \\ 0.75 & 1.00 & 0.60 \\ 0.30 & 0.30 & 0.40 \end{bmatrix}}$$

Matrix B shows the daily production capacity for these items at each outlet.

$$B = \begin{array}{l} \text{NSW} \\ \text{WA} \\ \text{SA} \end{array} \overset{\begin{array}{ccc}\text{shirt} & \text{blazer} & \text{jumper}\end{array}}{\begin{bmatrix} 20 & 12 & 24 \\ 18 & 16 & 18 \\ 24 & 10 & 22 \end{bmatrix}}$$

Matrix C summarises the hourly wages ($ per hour) of the different workers at each outlet.

$$C = \begin{array}{l} \text{cutting} \\ \text{sewing} \\ \text{finishing} \end{array} \overset{\begin{array}{ccc}\text{NSW} & \text{WA} & \text{SA}\end{array}}{\begin{bmatrix} 10.00 & 9.50 & 9.00 \\ 16.00 & 15.50 & 15.00 \\ 8.50 & 8.00 & 8.25 \end{bmatrix}}$$

(a) Determine the total time (in hours) required to cut, sew, and finish a blazer.

(1 mark)

(b) (i) Let $E = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Use matrix multiplication to calculate EA.

(1 mark)

(ii) Interpret your answer to part (b)(i).

(2 marks)

(c) (i) Let $F = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Use matrix multiplication to calculate AF.

(1 mark)

(ii) Interpret the first row of your answer to part (c)(i).

(1 mark)

(d) (i) Determine the maximum number of shirts, blazers, and jumpers that can be made daily by this company.

(2 marks)

(ii) How many days will it take this company to manufacture 450 blazers?

(1 mark)

(e) (i) Use matrix multiplication to calculate ACB.

(2 marks)

(ii) Interpret the value in row 1, column 2, of your answer to part (e)(i).

(1 mark)

You may write on this page if you need more space to finish your answers. Make sure to label each answer carefully (e.g. 'Question 3(a)(i) continued').

You may remove this page from the booklet by tearing along the perforations so that you can refer to it while you write your answers.

LIST OF MATHEMATICAL FORMULAE FOR USE IN STAGE 2 MATHEMATICAL METHODS

Standardised Normal Distribution

A measurement scale X is transformed into a standard scale Z using the formula

$$Z = \frac{X - \mu}{\sigma}$$

where μ is the population mean and σ is the standard deviation for the population distribution.

Confidence Interval — Mean

A 95% confidence interval for the mean μ of a normal population with standard deviation σ, based on a simple random sample of size n with sample mean $\bar{x}$, is

$$\bar{x} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{x} + 1.96\frac{\sigma}{\sqrt{n}}.$$

For suitably large samples, an approximate 95% confidence interval can be obtained by using the sample standard deviation s in place of σ.

Sample Size — Mean

The sample size n required to obtain a 95% confidence interval of width w for the mean of a normal population with standard deviation σ is

$$n = \left(\frac{2 \times 1.96\sigma}{w}\right)^2.$$

Confidence Interval — Population Proportion

An approximate 95% confidence interval for the population proportion p, based on a large simple random sample of size n with sample proportion

$\hat{p} = \frac{X}{n}$, is

$$\hat{p} - 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

Sample Size — Proportion

The sample size n required to obtain an approximate 95% confidence interval of approximate width w for a proportion is

$$n = \left(\frac{2 \times 1.96}{w}\right)^2 p^*\left(1 - p^*\right)$$

where p^* is a given preliminary value for the proportion.

Binomial Probability

$$P(X = k) = C_k^n p^k (1-p)^{n-k}$$

where p is the probability of a success in one trial and the possible values of X are $k = 0, 1, \ldots n$ and

$$C_k^n = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\ldots(n-k+1)}{k!}.$$

Binomial Mean and Standard Deviation

The mean and standard deviation of a binomial count X and a proportion of successes $\hat{p} = \frac{X}{n}$ are

$$\mu_X = np \qquad \mu(\hat{p}) = p$$

$$\sigma_X = \sqrt{np(1-p)} \qquad \sigma(\hat{p}) = \sqrt{\frac{p(1-p)}{n}}$$

where p is the probability of a success in one trial.

Derivatives

$f(x) = y$	$f'(x) = \frac{dy}{dx}$
x^n	nx^{n-1}
e^{kx}	ke^{kx}
$\ln x = \log_e x$	$\frac{1}{x}$

Properties of Derivatives

$$\frac{d}{dx}\{f(x) + g(x)\} = f'(x) + g'(x)$$

$$\frac{d}{dx}\{f(x) - g(x)\} = f'(x) - g'(x)$$

$$\frac{d}{dx}\{k f(x)\} = k f'(x)$$

$$\frac{d}{dx}\{f(x)g(x)\} = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Laws of Logarithms

$$\log A + \log B = \log AB$$

$$\log A - \log B = \log \frac{A}{B}$$

$$\log A^n = n \log A$$