CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the October/November 2013 series

9231 FURTHER MATHEMATICS

9231/13 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the October/November 2013 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a 'fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Marks	Total
1	Finds partial fractions.	$\dfrac{1}{r(r-1)(r+1)}=\dfrac{1}{2(r-1)}-\dfrac{1}{r}+\dfrac{1}{2(r+1)}$	B1	(1)	
	Expresses each term in fractions	$\left(\dfrac{1}{2}-\dfrac{1}{2}+\dfrac{1}{6}\right)+\left(\dfrac{1}{4}-\dfrac{1}{3}+\dfrac{1}{8}\right)\ldots\left(\dfrac{1}{2(n-1)}-\dfrac{1}{n}+\dfrac{1}{2(n+1)}\right)$	M1A1		
	Cancels terms and sums	$=\dfrac{1}{4}-\dfrac{1}{2n}+\dfrac{1}{2(n+1)}$ (OE)	M1A1	(4)	
	Find sums to infinity	$S\infty=\dfrac{1}{4}$	B1	(1)	[6]
2	Finds determinant and comments.	Determinant $=0\Rightarrow$ no inverse (AG)	M1A1	(2)	
	Eliminates one variable.	Obtains e.g. $x+y=1$	M1A1		
	Obtains general solution	$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\1\\-2\end{pmatrix}+t\begin{pmatrix}2\\-2\\3\end{pmatrix}$ (OE)	M1 A1	(4)	[6]
3	Finds complementary function	$m^2+2m+4=0\Rightarrow m=-1\pm\sqrt{3}i$ C.F. $\mathrm{e}^{-x}\left(A\cos\sqrt{3}x+B\sin\sqrt{3}x\right)$	M1 A1		
	Trial P.I.	P.I. $y=px^2+qx+r$ $y'=2px+q$ $y''=2p$			
	Differentiates twice and substitutes.	$2p+4px+2q+4px^2+4qx+4r=4x^2+8$	M1A1		
	Equates coefficients and solves	$\Rightarrow p=1, q=-1, r=2$	M1A1		
	States G.S.	$y=e^{-x}\left(A\cos\sqrt{3}x+B\sin\sqrt{3}x\right)+x^2-x+2$	A1	(7)	[7]

Qu No	Commentary	Solution	Marks	Part Marks	Total
4	Differentiates each equation wrt θ.	$\dot{x} = 2 - 2\cos 2\theta \quad \dot{y} = 2\sin 2\theta$	B1		
	Forms quotient for $\frac{\mathrm{d}y}{\mathrm{d}x}$	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\sin\theta\cos\theta}{4\sin^2\theta} = \cot\theta \quad \text{(AG)}$	M1A1		
	States exceptions.	$\text{Except for } \theta = k\pi \left(k \text{ an integer}\right).$	A1	**(4)**	
	Differentiates wrt x	$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}\theta}\cot\theta \times \frac{1}{\dot{x}} = -\operatorname{cosec}^2\theta \times \frac{1}{4\sin^2\theta}$	M1A1		
	Obtains result	$= -1 \text{ when } \theta = \frac{1}{4}\pi.$	A1	**(3)**	**[7]**
5	Uses sum of roots	$\sum \alpha = 3a = -\frac{36}{8} = -\frac{9}{2} \Rightarrow a = -\frac{3}{2}$	M1A1		
	Uses product of roots	$\alpha\beta\gamma = a\left(a^2 - d^2\right) = \frac{21}{8}$	M1		
	Substitutes for a,	$\frac{9}{4} - d^2 = -\frac{2}{3} \times \frac{21}{8} = -\frac{7}{4}$	M1		
	and solves	$\Rightarrow d^2 = 4 \Rightarrow d = \pm 2$	A1		
		Roots are $-\frac{7}{2}, -\frac{3}{2}, \frac{1}{2}$	A1		
	Uses sum of products in pairs.	$\sum \alpha\beta = \frac{21}{4} - \frac{7}{4} - \frac{3}{4} = \frac{k}{8}$	M1		
	(Or expands $(2x+7)(2x+3)(2x+1)$ and equates coefficients)	$\Rightarrow k = 22$	A1	**(8)**	**[8]**

Qu No	Commentary	Solution	Marks	Part Marks	Total
6					
(a)	Uses formula for MV and integrates	$\text{MV}=\dfrac{\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\sec x\,\mathrm{d}x}{\frac{1}{3}\pi-\frac{1}{6}\pi}=\dfrac{\left[\ln(\sec x+\tan x)\right]_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}}{\frac{1}{3}\pi-\frac{1}{6}\pi}$	M1A1		
	Substitutes limits	$=\dfrac{6}{\pi}\left\{\ln\left(2+\sqrt{3}\right)-\ln\left(\dfrac{2}{\sqrt{3}}+\dfrac{1}{\sqrt{3}}\right)\right\}$	M1		
	and evaluates	$=\dfrac{6}{\pi}\ln\left(\dfrac{2+\sqrt{3}}{\sqrt{3}}\right)$ (OE) (1.47)	A1	**(4)**	
(b)	Differentiates wrt x	$y=-\ln\cos x\Rightarrow y'=\tan x$	B1		
	Finds $\dfrac{\mathrm{d}s}{\mathrm{d}x}$	$\sqrt{1+(y')^2}=\sec x$	B1		
	Obtains arc length integral and integrates.	$s=\int_0^{\frac{\pi}{3}}\sec x\,dx=\left[\ln(\sec x+\tan x)\right]_0^{\frac{\pi}{3}}$	M1		
	Obtains result	$=\ln\left(2+\sqrt{3}\right)$ (1.32)	A1	**(4)**	**[8]**
7	States vertical asymptote	Vertical asymptote is $x=-2$	B1		
	Expresses y in a suitable form	$y=2x+1-3(x+2)^{-1}$	M1		
	States oblique asymptote	Oblique asymptote is $y=2x+1$	A1	**(3)**	
	Differentiates wrt x	$y'=2+3(x+2)^{-2}$	M1A1		
	and explains result	$(x+2)^{-2}>0\Rightarrow y'>2$ (AG)	A1	**(3)**	
	Sketches graph.	Axes and asymptotes	B1		
	(Deduct 1 mark for poor forms at infinity)	Each branch, showing intersection with axes.	B1B1	**(3)**	**[9]**

Qu No	Commentary	Solution	Marks	Part Marks	Total
8	Finds normal	$\overrightarrow{AB} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k},\ \overrightarrow{AC} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$			
	and cartesian equation	$\overrightarrow{AB} \times \overrightarrow{AC} = -6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$	M1A1		
		$6x + 2y - 5z = \text{constant} = 24 + 10 - 30 = 4$	M1A1	(4)	
	Finds where OD meets plane	Equation of OD: $\mathbf{r} = t(6\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$	B1		
		$\Rightarrow 36t + 6t - 30t = 12t = 4 \Rightarrow t = \frac{1}{3}$	M1A1		
		E is the point (2,1,2).	A1	(4)	
	Obtains angles between planes	Using $(6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}).\,2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$			
		$\Rightarrow 12 + 2 - 10 = \sqrt{36 + 4 + 25}\sqrt{4 + 1 + 4}\ \sin\theta$	M1A1		
		$\Rightarrow \theta = 9.5^\circ$ (0.166 rad).	A1	(3)	[11]
9	States inductive hypothesis	H_k: $(\cos\theta + \mathrm{i}\sin\theta)^k = \cos k\theta + i\sin k\theta$ for some k	B1		
	Proves $\mathrm{H}_k \Rightarrow \mathrm{H}_{k+1}$	$(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = (\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta)$	M1		
		$= (\cos\theta\cos k\theta - \sin\theta\sin k\theta) + \mathrm{i}(\sin\theta\cos k\theta + \cos\theta\sin k\theta)$	A1		
		$= \cos[k+1]\theta + 1\sin[k+1]\theta \quad \therefore \mathrm{H}_k \Rightarrow \mathrm{H}_{k+1}$	A1		
	States conclusion	H_1 is trivially true, so true for all positive integers	A1	(5)	
	Uses de M's Thm. to find $2\mathrm{i}\sin n\theta$	$z^n - \frac{1}{z^n} = (\cos n\theta + \mathrm{i}\sin n\theta) - (\cos n\theta - \mathrm{i}\sin n\theta)$ $= 2\mathrm{i}\sin n\theta$	B1		
	Uses binominal expansion	$\left(z - \frac{1}{z}\right)^5 = (2\mathrm{i}\sin\theta)^5 = 32\mathrm{i}\sin^5\theta$	B1		
	and groups	$= z^5 - 5z^3 + 10z - \frac{10}{z} + \frac{5}{z^3} - \frac{1}{z^5}$	M1		
		$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$	A1		
	Applies above result and obtains $\sin^5$	$32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$	M1		
		$\sin^5\theta = \frac{1}{16}(\sin^5\theta - 5\sin 3\theta + 10\sin\theta)$	A1	(6)	[11]

Qu No	Commentary	Solution	Marks	Part Marks	Total
10	Differentiates wrt θ	$\frac{\mathrm{d}r}{\mathrm{d}\theta} = 2\cos\theta - 2\cos 2\theta$	M1		
	Equates to zero	$2c - 2(2c^2 - 1) = 0 \Rightarrow 2c^2 - c - 1 = 0$	A1		
	And solves equation	$\Rightarrow (2c+1)(c-1) = 0$	M1		
		$\Rightarrow c = -\frac{1}{2}$ or 1	A1		
	States required points on C	$\left(\frac{3}{2}\sqrt{3}, \frac{2}{3}\pi\right)$	A1	**(5)**	
	Sketches C.	Approximate shape and location Accurate scaling.	B1 B1	**(2)**	
	Uses $\frac{1}{2}\int r^2 \mathrm{d}\theta$	Area $= \frac{1}{2}\int_0^{\frac{\pi}{4}} 4\sin^2\theta(1 - 2\cos\theta + \cos^2\theta)\mathrm{d}\theta$	M1 M1		
	Obtains an integrable form	$= \int_0^{\frac{\pi}{4}}\left(1 - \cos 2\theta - 4\cos\theta\sin^2\theta + \frac{1}{4}[1 - \cos 4\theta]\right)\mathrm{d}\theta$	A1A1		
	Integrates	$= \left[\frac{5\theta}{4} - \frac{\sin 2\theta}{2} - 4\frac{\sin^3\theta}{3} - \frac{\sin 4\theta}{16}\right]_0^{\frac{\pi}{4}}$	M1		
	Obtains result	$= \frac{5}{16}\pi - \frac{1}{2} - \frac{\sqrt{2}}{3}$ or 0.0103	A1	**(6)**	**[13]**

Qu No	Commentary	Solution	Marks	Part Marks	Total
11	**EITHER**				
	Integrates by parts	$I_n = \left[x\left(1+x^2\right)^n\right]_0^1 - 2\int_0^1 nx^2\left(1+x^2\right)^{n-1}\mathrm{d}x$	M1A1		
		$= 2^n - 2n\int_0^1\left(1+x^2-1\right)\left(1+x^2\right)^{n-1}\mathrm{d}x$	M1A1		
		$\Rightarrow I_n = 2^n - 2nI_n + 2nI_{n-1}$			
	Obtains reduction formula	$\Rightarrow (2n+1)I_n = 2nI_{n-1} + 2^n$ (AG)	A1	**(5)**	
	Finds I_0	$I_0 = \int_0^1 1\,\mathrm{d}x = 1$	B1		
	Uses reduction formula twice	$3I_1 = 2\times1+2 \Rightarrow I_1 = I_1 = \frac{4}{3}$	M1		
		$5I_2 = \frac{16}{3} + 4 \Rightarrow I_2 = \frac{28}{15}$	A1		
	Finds I_3	$7I_3 = \frac{56}{5} + 8 \Rightarrow I_3 = \frac{96}{35}$ (2.74)	A1	**(4)**	
	Uses reduction formula For n=-1 and n=-2	$2nI_{n-1} = (2n+1)I_n - 2^n$	M1		
	Eliminates I_{-2} and finds I_{-3}	$\Rightarrow 2I_{-2} = I_{-1} + \frac{1}{2}$ and $4I_{-3} = 3I_{-2} + \frac{1}{4}$	A1A1		
		$\Rightarrow I_{-3} = \frac{3}{8}I_{-1} + \frac{1}{4} = \frac{3}{32}\pi + \frac{1}{4}$	M1A1	**(5)**	**[14]**

Qu No	Commentary	Solution	Marks	Part Marks	Total
11	**OR**				
	States basic relationship (twice)	$\mathbf{Ae} = \lambda\mathbf{e}$ and $\mathbf{Be} = \mu\mathbf{e}$	B1		
	Adds	Adding $\mathbf{Ae} + \mathbf{Be} = \lambda\mathbf{e} + \mu\mathbf{e} \Rightarrow (\mathbf{A}+\mathbf{B})\mathbf{e} = (\lambda+\mu)\mathbf{e}$	M1		
	Obtains result	$\Rightarrow \lambda + \mu$ is an eigenvalue (since $\mathbf{e} \neq \mathbf{0}$)	A1	**(3)**	
	Finds eigenvalues of **A**	$\mathbf{A}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-1\end{pmatrix} \Rightarrow \lambda_1 = -1$	M1A1		
		$\mathbf{A}\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} \Rightarrow \lambda^2 = 1$	A1		
		$\mathbf{A}\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}6\\0\\3\end{pmatrix} \Rightarrow \lambda^3 = 3$	A1	**(4)**	
	States eigenvalues of **M**	Eigenvalues of **M** are $-8, -2, 2$	B1	**(1)**	
	States **R**	$\mathbf{R} = \begin{pmatrix}1 & 1 & 2\\1 & -1 & 0\\1 & 1 & 1\end{pmatrix}$	B1		
	Recognises that **S** is the inverse of **R**	$\mathbf{S} = \mathbf{R}^{-1}$	M1		
	Finds **S**	Determinant = 2 $\mathbf{S} = \frac{1}{2}\begin{pmatrix}-1 & 1 & 2\\-1 & -1 & 2\\2 & 0 & -2\end{pmatrix}$	M1A1		
	Finds **D**	$\mathbf{D} = \begin{pmatrix}-32768 & 0 & 0\\0 & -32 & 0\\0 & 0 & 32\end{pmatrix}$	M1A1	**(6)**	**[14]**