FURTHER MATHEMATICS

Paper 9231/01
Paper 1

General comments

The scripts for this paper were of a generally good quality. There were a number of outstanding scripts and many showing evidence of sound learning. Work was well presented by the vast majority of candidates. Solutions were set out in a clear and logical order. The standard of numerical accuracy was good. Algebraic manipulation, where required, was sound. The improvement in vector work continued.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. A very high proportion of scripts had substantial attempts at all twelve questions. Once again there were few misreads and few rubric infringements.

Candidates displayed a sound knowledge of most topics on the syllabus. As well as the vector work, already mentioned, candidates tackled the questions on matrices, differential equations, roots of equations and polar coordinates confidently. The questions on linear algebra and proof by induction caused the most difficulty.

Comments on specific questions

Question 1

There were many completely correct answers to this question. Those who were unable to complete it needed to simplify $1+\left(\frac{1}{2}\left[e^{2x}-e^{-2x}\right]^2\right)$ to $\frac{1}{4}\left[e^{2x}+e^{-2x}\right]^2$.

Question 2

This question was well done by many candidates. Almost all could find the correct partial fractions and knew to use them with the method of differences. There were some who cancelled terms incorrectly. In this case it was often possible to earn the mark for the sum to infinity on a follow through basis.

Answers: $S_N = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right]$; $\frac{3}{4}$.

Question 3

There were many correct answers to this question. Almost all candidates knew the correct formula for the y-coordinate of the centroid of R. The denominator was usually found correctly and most errors often occurred in the final calculation, having integrated y^2 correctly in the numerator. A small number of candidates lost the final mark by giving a decimal answer.

Answer: $\frac{3}{8}\ln 2 + \frac{9}{32}$ or any equivalent form.

Question 4

The same difficulties arose with this proof by induction as have occurred in similar questions in the past. In addition, in this question, the proof was required for *all non-negative integers*, hence the base case was $n = 0$ and not $n = 1$. The major difficulty was being able to prove $H_k \Rightarrow H_{k+1}$. Often the key idea was to think of 7^{2k+3} as $7^2.7^{2k+1}$ rather than $7^3.7^{2k}$. The proof was completed in a variety of ways; e.g. taking $\mathrm{f}(n) = 7^{2n+1} + 5^{n+3}$, some considered $\mathrm{f}(n+1)$ itself, or $\mathrm{f}(n+1) - \mathrm{f}(n)$, and used the fact that 7^2 could be written as 44 + 5, while others showed that $\mathrm{f}(n+1) - 5\mathrm{f}(n) = 308.(7^{2k})$ and that 44 was a factor of 308.

Question 5

Many candidates were able to prove the reduction formula by integrating by parts twice. There were occasional sign errors and some candidates succeeded by taking more than one attempt to get a correct solution. In the evaluation of I_6, there were some errors in evaluating I_0. Those who found, in sequence, I_0, I_2, I_4, I_6 sometimes made a small arithmetical error in finding I_4, leading to a more substantial error for I_6. Others preferred to find I_6 in terms of I_0 and then substitute for I_0. Both methods appeared equally successful.

Answer: 0.0177.

Question 6

The first part of this question was done well by nearly all candidates. The second part caused considerable difficulty to most. Invariably candidates thought that they were being asked to find a basis for the range space of T and produced a well-rehearsed method. Candidates needed to realise that the question gave them the basis and asked them to show that it was a basis. This required them to show that the three vectors were linearly independent, and then to observe that, since dim R(T) was not 4 (from the first part), then they did indeed have a basis for the range space of T. In the final part, many wrote down the four correct equations. Only a few managed to obtain the condition satisfied by *p* and *q*. Possible approaches were to find each of *a*, *b* and *c* in terms of *p* and *q*, and then substitute these in the fourth equation, or to eliminate successively *a*, *b* and *c*, finally leaving one equation in *p* and *q*, or to work with row operations, as in the first part, using the augmented matrix.

Answer: $6p + q = 3$.

Question 7

This question was very well done by the vast majority of candidates. There were only a few cases where the algebra went wrong in the first part. The numerical values for the cases *n* = 1 and *n* = 2 were usually obtained. The case of *n* = 3 was done, usually by substituting each root in turn and summing to obtain $6S_3 = 7S_2 - 3S_2 + 3$, or by using a formula such as $S_3 = \left(\sum \alpha\right)^3 - 3\sum \alpha \sum \alpha\beta + 3\alpha\beta\gamma$. In the final part, many saw that the left-hand side of the displayed result could be expressed as

$(\alpha+1)(\beta+1)(\gamma+1)\left\{\frac{1}{(\alpha+1)^3} + \frac{1}{(\beta+1)^3} + \frac{1}{(\gamma+1)^3}\right\}$ and could thus obtain $\frac{73}{36}$.

Answers: $\frac{7}{6}$; $\frac{13}{36}$; $\frac{73}{216}$.

Question 8

This question was approached in a generally correct manner. In part **(i)** the point $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$ was frequently omitted. There were some instances of r and θ being given in the wrong order. The graphs in part **(ii)** were mostly correct. Sometimes they lacked smoothness, but this was only penalised in cases of severe distortion. Occasionally the part of the cardioid below the level of the initial line was missing. The calculation of the area in part **(iii)** was mostly correct. The method for integrating $\sin^2\theta$ was well known. Common errors were the omission of the $\frac{1}{2}$ factor in the area formula and incorrect limits for integration.

Answers: **(i)** $\left(\frac{3}{2}, \frac{\pi}{6}\right)$, $\left(\frac{3}{2}, \frac{5\pi}{6}\right)$.

Question 9

This question proved to be a good source of marks for many candidates. The characteristic equation and eigenvalues were usually found correctly. Almost all candidates knew a method for finding corresponding eigenvectors and were able to find at least one correctly; frequently two, or all three, were correct. The final part was done well and candidates could earn all 3 marks on a follow through basis, when earlier errors had occurred. Sometimes candidates forgot to cube values in matrix **D**.

Answers: 1, 3, 4; $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$; $\mathbf{M} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}$; $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

Question 10

The proof of the identity for $\tan 5\theta$ was well done by most candidates. Because the result was displayed, some working was required to demonstrate how an expression in terms of sines and cosines could be changed to one in terms of tangents. The vast majority of candidates did this well, showing good examination technique when displayed results are given. The remainder of the question caused considerable difficulty for a substantial number of candidates. Candidates needed to observe that $\tan 5\theta = 0$. This led to $t(t^4 - 10t^2 + 5) = 0$. Some justification was required for the values of n in the displayed result, rather than merely stating them. In the final part it was necessary to regard the equation as a quadratic in $x^2\ (= \tan^2\theta)$, then consideration of the signs of $\tan\left(\frac{1}{5}n\pi\right)$ gave the required value from the product of the roots of the equation.

Answer: $\sqrt{5}$.

Question 11

The initial proof was done well by large numbers of candidates. It was considerably easier to differentiate the given expression for z twice, rather than to re-write the given expression as y in terms of z and x, before differentiating twice. Those who did the latter were able to succeed, but had to be careful with negative indices. Complementary functions and particular integrals were mostly found correctly, and added, to give the general solution. The coefficient of $\cos 2x$ was often found correctly by substituting boundary conditions in y. The coefficient of $\sin 2x$ was often inaccurate, although a correct method was attempted. This led to the loss of the final 2 marks.

Answer: $y = \frac{1}{x}\left(\left(\frac{\pi^2}{2} + 3\right)\cos 2x + \frac{3\pi}{2}\sin 2x\right) + 2x + \frac{3}{x}$.

Question 12 EITHER

There were many good answers to the first four parts of this question. In part **(i)** almost all attempted to differentiate and put the gradient function equal to zero, thus producing a quadratic equation in x. Even when the quadratic had incorrect coefficients, a mark could still be earned for stating that there were at most (not exactly) two stationary points. In part **(ii)** most used the discriminant correctly to obtain an inequality, but inaccurate coefficients in part **(i)** meant the loss of a further mark here. Many saw that part **(iii)** required the numerator, in the expression for y, to be zero, and that using the discriminant for this quadratic equation gave the required condition. Almost all attempting this question could do part **(iv)(a)** accurately, and most, realising that the horizontal asymptote was $y = 1$, solved the resulting equation correctly. The attempts at graphs generally left room for improvement. The first graph needed two vertical asymptotes, no turning points, the middle branch to pass through the origin and the outside branches to approach the horizontal asymptote from above, for negative values of x, and from below, for positive values of x. The second graph had two turning points, a maximum for some value of $x > 0$ and a minimum for some value of x such that $-2\lambda < x < 0$. The graph again passed through the origin and approached the horizontal asymptote from below, for negative values of x, and from above, for positive values of x. Each graph crossed the horizontal asymptote once.

Answers: **(iii)** $\lambda < 1$, **(iv)(a)** 0, -2λ, **(b)** $\dfrac{\lambda}{2(\lambda+1)}$.

Question 12 OR

This alternative was a little more popular than the other. The improved work on vectors, noticed recently, continued. Almost all attempting this question could find a cartesian equation of Π_1 correctly, usually using a vector product, while others eliminated λ and μ from the parametric equations. The direction of the line of intersection of Π_1 and Π_2 was often found by using a vector product. Only a few candidates doing this were correctly able to find a point on the line. A more secure method was to eliminate one variable from the two cartesian equations, then, using a parameter, express all three variables in terms of the parameter, thus giving an equation of the line. The distance of a point from a plane formula was usually used to find the value of a. A few sign errors crept in and sometimes the condition $a > 0$ was ignored. In the final part an appropriate scalar product was formed, but too often it was equated to $\dfrac{1}{\sqrt{3}}$ rather than $\dfrac{2}{\sqrt{6}}$. Similarly, here, $c > 0$ was ignored. Negative values in these final two parts were penalised only once in the question.

Answers: $x - 2y + z = 4$; $\mathbf{r} = -4\mathbf{i} - 4\mathbf{j} + t(3\mathbf{i} + 2\mathbf{j} + \mathbf{k})$; 2, 2.

FURTHER MATHEMATICS

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Paper 2

General comments

Most candidates attempted all the questions, and some very good answers were seen. In the only question which offered a choice, namely **Question 11**, there was a preference for the Statistics alternative, though the Mechanics alternative also produced good attempts from some candidates. Indeed all questions were answered well by some candidates, most frequently **Questions 1** and **8**. Conversely **Questions 4** and **9** appeared to be generally the most challenging.

Candidates' working was often set out sufficiently clearly, with any corrections legible and the replacements to deleted attempts readily identifiable, though there was room for improvement in some cases. In some of the Mechanics questions it is helpful to include diagrams and candidates are encouraged to do so, for example to show the forces in **Question 4**. This question affords a particularly good example of the need to explain working clearly, since it requires candidates to show that certain given results are true rather than finding unknown results. In these circumstances it is advisable to justify any equations which are written down, for example by stating that forces are being resolved in a specified direction or moments are being taken about a specified point. Even when an unknown result must be found, such as a numerical value in a Statistics question, candidates are well advised to show their method, since credit may then be earned for a valid approach if, for example, an incorrect result is due only to an arithmetical error.

Comments on specific questions

Question 1

The value $\frac{3}{2}$ of the SHM parameter ω is first found from the standard expression $v^2 = \omega^2(x_0^2 - x^2)$ since the values of x, x_0 and v are stated in the question, and the period then follows from $\frac{2\pi}{\omega}$. The required maximum speed and acceleration may then be found by calculating ωx_0 and $\omega^2 x_0$, and most candidates had little difficulty in doing so.

Answers: $\frac{4\pi}{3}$ or 4.19 s; **(i)** 7.5 ms^{-1}; **(ii)** 11.25 ms^{-2}.

Question 2

The validity of the given equation was readily shown by almost all candidates using conservation of energy. Most went on to correctly equate the radial forces on the particle, showing that the contact force is $mg\cos\theta + \frac{m\left(\frac{1}{2}u\right)^2}{a}$, apart from an occasional sign error. A careful reading of the question shows that this is not in the required form since it involves θ, which may be remedied by employing the initial result to replace θ.

Answer: $mg - \frac{mu^2}{8a}$.

Question 3

By applying conservation of momentum and Newton's restitution equation, the unknown speed of *B* after its collision with *A* may be eliminated in order to find the required coefficient of restitution *e*. Care needs to be taken over the signs of the various terms and in the elimination process, and of course in the restitution equation *e* must multiply the difference of velocities before rather than after the collision. The required inequality follows immediately from $e < 1$, and candidates who cannot obtain the inequality should be alerted by this to a probable error in their expression for *e*.

Answer: $\frac{4+\alpha}{3\alpha}$.

Question 4

Candidates should ideally begin questions such as this by identifying all the relevant forces and their directions if known, consisting in this case of the contact force at *A* acting towards *C*, the contact force at *B* acting normal to the rod, and the weight of the rod. Since a variety of resolutions and moment equations are possible, it is advisable to choose those which will yield the required unknowns most readily, thereby minimising the amount of work involved and reducing the probability of error. Here a resolution of forces parallel to the rod gives the contact force at *A* immediately, and that at *B* then follows from resolution normal to the rod. The final given result follows most easily from taking moments about *A* once the contact force at *B* has been found, with moments about any other point requiring a little more work. As mentioned earlier, the proof of the required results is more convincing if the origin of all resolution and moment equations is explained.

Question 5

Candidates should be aware that, despite the *List of Formulae* including the moments of inertia for a lamina or disc only about a perpendicular axis, axes which are not perpendicular may also be encountered. This is the case here, where the axis of rotation is said to be tangential to the disc rather than perpendicular to it. To find the moment of inertia about this axis at *A*, the perpendicular axis theorem can first be used to show that the moment of inertia of the disc about a diameter is half that about a perpendicular axis through the centre and thus equals $\frac{1}{2}ma^2$, and then the parallel axes theorem gives the moment of inertia about a tangential axis through *A* as $\frac{5ma^2}{2}$. Addition of the moment of inertia of the particle yields the given result *I* for the system. Most candidates rightly equated the angular kinetic energy of the system to the change in potential energy as *AB* rotates from the horizontal to the vertical, but care is needed in finding the latter energy. One method is to find the distance fallen by the centre of gravity of the system, whose total mass is $3m$, but it is probably a little easier to sum the separate changes in potential energy for the disc and particle, respectively $2mg \times a$ and $mg \times 2a$. Similar care is needed in finding the couple in the equation of motion, after which the approximate period is readily found by approximating $\sin\theta$ by θ and then using the usual formula $T = \frac{2\pi}{\omega}$.

Answers: $\sqrt{\frac{16g}{13a}}$; $I\frac{d^2\theta}{dt^2} = -4mga\sin\theta$; $2\pi\sqrt{\frac{13a}{8g}}$.

Question 6

In order to answer this question, candidates must select the appropriate formula for the semi-width of the required confidence interval, here $z\sigma\sqrt{(\frac{1}{15} + \frac{1}{20})}$. The value of the standard deviation σ is given in the question as 15, the appropriate tabular value of *z* is 1.645, and the difference in sample means is clearly 3, leading to the required result.

Answer: [–5.43, 11.4].

Question 7

Almost all candidates deduced that the usual parameter p is here $\frac{1}{4}$, so that $q = \frac{3}{4}$, and made a reasonable attempt at writing down or finding appropriate expressions involving p and/or q for the three probabilities. These are respectively q^4p, q^4 and $1 - q^N$, or their equivalent. Care is needed when handling inequalities between negative values, as occurs after taking logarithms of each side in the final part, since there is a risk of wrongly concluding that $N < 26.4$ rather than $N > 26.4$.

Answers: **(i)** 0.0791; **(ii)** 0.316; **(iii)** 27.

Question 8

The contingency table of expected values was usually found correctly, as was the calculated χ^2-value of about 3.68, the accuracy of this result depending on how many significant figures were employed during the calculation. This must be compared with the tabular value 5.99, leading to acceptance of the null hypothesis. The question indicates that the required test concerns a possible association, but candidates need to formulate a correct statement of the null hypothesis to be tested. This is that no association exists, or equivalently that passing or failing the driving test is independent of the driving school attended, rather than the opposite.

Question 9

The key to answering this question correctly is the selection of the most suitable test statistic. In this case a paired sample t-test is most appropriate, so candidates should base their test on the differences between the two sample times for each of the eight runners, under the explicit assumption that the population of the differences has a normal distribution. The less appropriate two-sample t-test requires that the two populations of indoor and outdoor times are each normally distributed and also that they share a common variance. In the paired-sample test the sample mean is found to be 0.225 and the unbiased estimate s^2 of the population variance is found to be 0.0164, giving a t-value of 2.76. When finding this latter value, candidates must recall that their stated hypotheses concern whether or not the mean of the population of differences exceeds 0.1 rather than 0, so that t is calculated from $\dfrac{0.225 - 0.1}{s\sqrt{\frac{1}{8}}}$. Since the corresponding tabular t-value is 2.365, the null hypothesis is rejected, confirming the coach's suspicion.

Question 10

Calculation of the mean values, the gradient and hence the equation of the regression line and the product moment correlation coefficient r are all straightforward, with relevant expressions given in the *List of Formulae*. Apart from an occasional arithmetical error most candidates experienced little difficulty. Commenting sensibly on the value of the correlation coefficient is more challenging, since candidates should note that the question requires their comment to be in context, and thus a vague and/or generalised statement is insufficient. The final test requires an explicit statement of the null and alternative hypotheses, $\rho = 0$ and $\rho < 0$, and here candidates should be aware that r and ρ are not the same variable. Finally comparison of the magnitude of the previously calculated value of r with the tabular value 0.658 leads to a conclusion of there being evidence of negative correlation.

Answers: **(i)** 2.02, 3.82; **(ii)** –4.38; **(iii)** $y = 12.7 - 4.38x$; **(iv)** –0.719.

Question 11 EITHER

Although less popular than the Statistics alternative discussed below, some candidates made very good attempts at this alternative, with only the final part causing frequent difficulty. The given position of the equilibrium point *O* is readily verified by equating the net effect of the tensions in the two elastic strings to zero. The given equation of motion is then also found by using the difference of the two opposing tensions, but at the general point corresponding to a displacement *y* from *O*. The question specifies that *y* is measured in the direction *OB*, but the equation of motion is of course the same for the reverse direction. Noting that the motion is simple harmonic with $\omega^2 = 500$, the period is readily found from $\frac{2\pi}{\omega}$. Another standard SHM result enables the required speed to be found from $\omega\sqrt{(0.2^2 - 0.1^2)}$. Although most candidates realised that the required time in the final part can be found by an appropriate application of an SHM formula such as $x = 0.2 \sin \omega t$ or $x = 0.2 \cos \omega t$, particular care is needed since the particle passes through the equilibrium point before reaching the specified position for the first time. The simplest route to the answer is probably to calculate the term $\frac{1}{\omega}\cos^{-1}\left(\frac{-0.1}{0.2}\right)$, since alternative expressions usually require the incorporation of an appropriate fraction of the period.

Answers: 0.281 s; **(i)** 3.87 ms^{-1}; **(ii)** 0.0937 s.

Question 11 OR

Finding the distribution function was widely known to require integration of f(*t*), but candidates must either find the definite integral from 0 to *t* or else the indefinite integral with a constant of integration then determined from, say, F(0) = 0. Deducing that $\lambda = \frac{1}{20}$ and hence evaluating F(15) presented little difficulty, and most candidates used the table of values of F(*t*) to calculate the required expected values correctly by differencing adjacent entries and multiplying by 100. Since each expected value used in the χ^2-test must be at least 5, candidates need to combine the two adjacent intervals with expected values 4.93 and 3.85. The χ^2-test is then conducted in the usual way, and comparison of 3.58 with the tabular value 14.07 leads to acceptance of the null hypothesis, namely that the specified exponential distribution is indeed a suitable model.

Answers: 0.5276; 22.12, 17.23, 13.41, 10.45, 8.14, 6.34, 4.93, 3.85, 13.53.