FURTHER MATHEMATICS

Paper 9231/01

Paper 1

General comments

The quality of work received in response to this examination showed a pleasing improvement on that received for the corresponding examination last November. There were fewer very poor scripts and a good number of outstanding ones. The better candidates continued to present their work well and the weaker candidates showed some improvement in presentation on what was produced last year. The majority of candidates were able to cope with the calculus required on this paper, but algebraic technique remained a problem for a sizeable minority. Numerical accuracy was mostly of a good standard.

There was no evidence to suggest that candidates had any difficulty in completing the paper in the time allowed. Most candidates made substantial attempts at all the questions. There continued to be few misreads and infringements of the rubric.

Many candidates appeared to have been well prepared on all parts of the syllabus. The work on complex numbers, linear spaces and induction was a considerable improvement on previous offerings in these parts of the syllabus.

Comments on specific questions

Question 1

Almost all candidates could find the derivatives of x and y with respect to t. Most of these were then able to apply the formula for arc length. There were a disappointing number of candidates who could not do the necessary algebra in order to square and add the two results and produce a perfect square. Performance on this question was often a good indicator of performance on the paper as a whole.

Question 2

With the exception of the small number of candidates who did not know how to calculate the mean value of a function over a range, most were able to complete the first part of the question correctly. A large number were able to make progress on the second part, usually by evaluating $\int_1^{e^2} \ln y \, dy$. A number of good candidates, by sketching a graph, realised that the area found in the first part of the question could be subtracted from $2e^2$ and the result immediately followed.

Answer: $\frac{1}{2}\left(e^2 - 1\right)$ or 3.19.

Question 3

Most candidates were able to sketch the polar curve with approximately the correct shape and location. Some, however, did not have a curve that was tangential to the half line $\theta = \frac{\pi}{2}$ at the pole, or did not have a negative gradient at the point $\left(\frac{\pi^2}{4}, 0\right)$.

The majority of candidates could right down the appropriate integral for finding the area of the region bounded by *C* and the initial line. Errors frequently occurred when candidates tried to expand the brackets before integrating and occasionally candidates dropped a negative sign when integrating without expanding brackets.

Answer: $\frac{\pi^5}{320}$.

Question 4

Many candidates were unable to tackle the initial proof at all. Some could only write the opening line: $\mathbf{Ae} = \lambda\mathbf{e}$. Of the remaining candidates, about half went wrong with the algebra and half successfully completed the proof. Candidates were more successful with the calculation, but some omitted the 2 in the 2**I** in their addition.

Answer: 110.

Question 5

The first part of the question required correct working only. Since the value of *y* was zero at the point *A*, a number of candidates obtained the result and did not realise that they had made an error. If the error was minor they could gain both method marks and possibly one accuracy mark in the second part of the question. The second differentiation was done well by many candidates and there were a high number of completely correct answers to this question.

Answer: – 9.

Question 6

The vast majority of candidates knew that it was necessary to reduce the matrix **A** to echelon form and invariably this was done correctly. Those who worked with α rather than 9 were able to use their reduced matrix in the final part of the question. Finding the basis for the null space of **A** caused problems for some. The usual method was to find a two-parameter solution for the two equations and separate the terms. There were six essentially different acceptable results, each of which could have opposite signs, or constant multipliers.

Answers: **(i)** $\left\{\begin{pmatrix}5\\3\\1\\0\end{pmatrix},\begin{pmatrix}1\\4\\0\\-1\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}17\\0\\4\\3\end{pmatrix},\begin{pmatrix}5\\3\\1\\0\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}17\\0\\4\\3\end{pmatrix},\begin{pmatrix}1\\4\\0\\-1\end{pmatrix}\right\}$ or

$\left\{\begin{pmatrix}5\\3\\1\\0\end{pmatrix},\begin{pmatrix}0\\-17\\1\\5\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}1\\4\\0\\-1\end{pmatrix},\begin{pmatrix}0\\-17\\1\\5\end{pmatrix}\right\}$ or $\left\{\begin{pmatrix}17\\0\\4\\3\end{pmatrix},\begin{pmatrix}0\\-17\\1\\5\end{pmatrix}\right\}$;

(ii) 3.

Question 7

The majority of candidates could successfully differentiate the given expression and although they realised it was required to integrate the result, with respect to x, between 0 and 1, few were able to do the necessary manipulation in order to achieve the reduction formula. The key to doing this is to realise that

$$\frac{4nx^4}{(1+x^4)^{n+1}} = 4n\frac{[(1+x^4)-1]}{(1+x^4)^{n+1}} = \frac{4n}{(1+x^4)^n} - \frac{4n}{(1+x^4)^{n+1}}$$ or equivalent.

Questions requiring this technique have frequently been set in the past, but these candidates seemed largely unaware of it. Most candidates were able to successfully use the reduction formula in the required manner.

Answer: 0.710.

Question 8

There were many complete and accurate answers to this question. Most candidates could solve the auxiliary equation and write the complementary function correctly. Most knew the correct form for the particular integral, but algebraic errors occurred among the weaker candidates. The values of the arbitrary constants were correctly found by many, but some failed to differentiate the general solution and so could not obtain the value of the second arbitrary constant.

Answer: $y = -\frac{1}{4}e^{-\frac{3t}{5}}\left\{4\cos\left(\frac{4t}{5}\right) + 3\sin\left(\frac{4t}{5}\right)\right\} + t^2 + 1$.

Question 9

A small number of candidates did not obtain the result by induction and so could gain no marks on the first part of the question, since there was no ‘or otherwise’ invitation. Those who attempted induction were invariably able to prove the base case and state the inductive hypothesis correctly. They then added the $(k + 1)$th term to the result for $n = k$ and mostly achieved the correct result, although weaker candidates, as elsewhere in the paper, made errors with the algebra. There were a considerable number of candidates who did not write a satisfactory conclusion to the proof, even when they had scored all the preceding marks. A statement such as: H_1 is true and $H_k \Rightarrow H_{k+1}$, so, by the principle of mathematical induction, H_n is true for all positive integers n, would suffice. A pleasing number of candidates were able to find the sum of the series for $n = N + 1$ to $n = 2N$ and many of these were able to prove the inequality.

Question 10

Many candidates were able to expand $(\cos\theta + i\sin\theta)^n$ by the binomial theorem and equate the real part to $\cos 8\theta$. Most of them were then able to replace $\sin^2\theta$ by $1-\cos^2\theta$ and obtain a polynomial in $\cos\theta$ as required. Weaker candidates again found trouble getting the algebra correct and made errors with signs and coefficients. Most then replaced $\cos^2\theta$ by $1-\sin^2\theta$ in order to obtain the polynomial in $\sin\theta$. Only the better candidates realised that a substitution of $\theta \to \frac{\pi}{2} - \theta$ would achieve the same result, only more easily.

The better candidates also were able to combine $x = \cos^2\frac{\pi}{8}$ and the identity in order to evaluate the given expression. The stumbling block for weaker candidates seemed to be the realisation that $\cos\left(8\times\frac{\pi}{8}\right) = \cos\pi = -1$.

Answers: **(i)** $128\cos^8\theta - 256\cos^6\theta + 160\cos^4\theta - 32\cos^2\theta + 1$; **(ii)** $-\frac{1}{16}$.

Question 11

In the first part of the question it was necessary to show the appropriate vector product to obtain the normal vector. Working backwards from the printed answer received no credit. Since the answer was printed, it was necessary to show some working in order to calculate the constant term in the plane equation. Very few candidates were able to make progress on the middle part of the question. Those who did so frequently omitted a modulus sign and even those who included it ended with a one-sided inequality. Others interpreted 'not greater than' as 'less than'. There were many complete and accurate answers to the final part of the question, with only a small number failing to calculate the vector 24**i** – 6**j** –12**k** at the start of this part.

Answers: –2**i** –3**j** – 6**k**; $-1 \le \lambda \le 0$; 77.4º.

Question 12 EITHER

Parts **(i)** and **(ii)** were invariably done correctly. In part **(iii)** most candidates obtained the correct quadratic equation. They then used the discriminant to find the range of values for *a*, but unfortunately ignored the information, given in the question, which prohibited *a* from being 1,2 or 3, when stating the final inequality. There were few completely correct graphs. Part **(iv)(a)** was more often correct than part **(iv)(b)**. In the latter, the middle branch either had no local maximum point, or if it did, it was too low. The right-hand branch seldom had any stationary point at all.

Answers: **(i)** $x = 1$, $x = 3$, $y = 1$; **(iii)** $1 < a < 3$ ($a \neq 2$).

Question 12 OR

The initial proof caused difficulty for many of those attempting this alternative. They did not realise the need to multiply the equation by x^n before substituting each of the roots in turn and summing. Those who realised this had no difficulty in completing the proof. Candidates who chose to do this alternative, in the main, knew the correct method for each of the three parts. A considerable number, however, made errors with the numbers. Often this was because they incorrectly calculated S_0 or S_{-1} at an early stage, resulting in the loss of further accuracy marks.

Answers: **(ii)** 10, 54; **(iii)** –6, 292; **(iv)** 248.

FURTHER MATHEMATICS

Paper 9231/02
Paper 2

General comments

The quality of the candidates' work varied very greatly, so in that sense the paper discriminated well. The paper did not appear to be too long for the time allowed, since the better candidates were able to make reasonably complete attempts at all questions. In general the Statistics questions were answered more successfully, and in the single question offering a choice, namely **Question 11**, the Statistics alternative attracted many more attempts. Although as usual some questions were found by the candidates to be more demanding than others, none was found to be unduly difficult or easy.

Comments on specific questions

Question 1

Most successful attempts considered each of the three sections of wire in turn, using the formula for the moment of inertia about the Centre of a thin rod given in the List of Formulae together with the parallel axis theorem to find the moment of inertia of each section about *A*. Some candidates were instead able to write down the moment of inertia about *A* directly for the sections *AB* and *AC* even though this is not included in the List of Formulae. Finally the three moments are summed. A common mistake was to use *m* as the mass of each of the three sections instead of calculating the appropriate fraction of *m*, while a few candidates erroneously based their attempt on the moment of inertia of a rectangular lamina with diagonal *AC* of which the triangle *ABC* supposedly constitutes half.

Answer: $36ma^2$.

Question 2

By considering conservation of energy between the highest point *A* and each of the two specified positions B_1 and B_2 of the bead at θ and $\theta + \pi$, the speeds at B_1 and B_2 may be found (note that conservation of energy between B_1 and B_2 is insufficient). These speeds may then be substituted into the appropriate equation at each point resulting from a resolution of radial forces in order to give the force magnitudes R_1 and R_2 in terms of θ. Subtraction then yields the given result. Some candidates made sign errors, particularly at the first point, while others introduced $\cos(\theta + \pi)$ at B_2 but did not relate it to $\cos \theta$ correctly. Instead of general points, a few candidates wrongly considered the specific points at which $\cos \theta = \frac{2}{3}$.

Question 3

The key to finding the angular acceleration $\frac{d^2\theta}{dt^2}$ of the disc correctly is to realise that the linear acceleration $a\frac{d^2\theta}{dt^2}$ of the block is dependent on the tension T as well as on the forces $2mg$ and $\frac{mg}{10}$. A second equation involving T comes from $I\frac{d^2\theta}{dt^2} = aT$, where the moment of inertia I of the disc is $\frac{1}{2}ma^2$, and eliminating T between these two equations yields the required angular acceleration. Instead a very large number of candidates wrongly took the couple producing the angular acceleration of the disc to be just $a(2mg - \frac{mg}{10})$. The required angular speed when $\theta = 2\pi$ is obtained from $\sqrt{(2\theta\frac{d^2\theta}{dt^2})}$.

Answers: $\frac{7.6}{a}$; $\frac{9.77}{\sqrt{a}}$.

Question 4

Almost all candidates correctly applied the conservation of momentum to the collision in order to produce an expression for m, although a few introduced a sign error. Showing that $m < 0.25$ proved more of a challenge, however, causing many candidates to consider only the special case in which both particles are at rest after the collision. A valid argument follows from noting that if A does not change direction then, since B cannot pass through A, it must by contrast change direction. The required inequality then follows either from the expression for m or from the original momentum equation. In the second part, application of Newton's restitution equation to the collision yields a second equation involving v_A and v_B, enabling either to be found by combining the two equations, and hence the required impulse.

Answers: **(i)** $m = \frac{0.5 - 0.1v_A}{2 + v_B}$; **(ii)** 0.7.

Question 5

Candidates seemed to find this the most difficult of the compulsory Mechanics questions. The starting point is to equate $m\frac{d^2x}{dt^2}$ to the net force on the particle, found by replacing PA and PB by $a + x$ and $a - x$ respectively, though many candidates did not even reach this point. The factors $\left(\frac{PA}{a}\right)^{-\frac{1}{4}}$ and $\left(\frac{PB}{a}\right)^{\frac{1}{2}}$ may then be approximated by $1-\frac{x}{2a}$ and $1-\frac{x}{4a}$ respectively, yielding the standard SHM equation with here $\omega^2 = \frac{g}{4a}$. The required speed v_{max} of P at the mid-point of the motion is then ωx_{max}, where the amplitude x_{max} is here $0.04a$. Another standard SHM formula for the speed v after time t, namely $v = x_{max}\,\omega \sin \omega t$, may then be employed with $v = \frac{1}{2}v_{max}$ in order to find the required time even if v_{max} has not been determined correctly. A common error in the final part was to unthinkingly use cos instead of sin in the expression for v.

Answers: $0.0632\sqrt{a}$; $0.331\sqrt{a}$.

Question 6

Almost all candidates used the appropriate standard formula and found the pooled estimate of σ^2 correctly.

Answer: 71.8.

Question 7

Apart from the occasional use of the wrong tabular *t*-value instead of 2.492, the sample mean pulse rate $\bar{x}$ and the standard deviation s were often found correctly by substitution into the formula for the confidence interval limits $\bar{x} \pm \frac{ts}{n}$. The required assumption of the population having a normal distribution was only stated by a minority of the candidates, some of whom instead referred to the normality of the sample. While it may be concluded that UK adult males who exercise have a reduced pulse rate, this is because the mean pulse rate for all UK adult males is well outside the stated confidence interval, and not simply because this mean pulse rate is above the estimated pulse rate of those who exercise.

Answers: **(i)** 62.8, 3.19.

Question 8

Most candidates performed well on this question, except for realising that the product moment correlation coefficient for the sample is negative. The sample means follow from noting that they satisfy the two given equations for the regression lines, and solving them. A number of candidates simply solved the equations for variables x and y rather the sample means, without explanation, which cast some doubt on whether they fully understood the relationship between the sample means and the regression lines. The correlation coefficient r is found from $\sqrt{(0.425 \times 0.516)}$, but is clearly negative. The hypotheses in the final part are best expressed using a different symbol than r, $\rho = 0$ and $\rho \neq 0$ say, since r is essentially an estimate of ρ derived from the sample. Since the magnitude of r is greater than the relevant tabular value 0.396, the test leads to the conclusion that the population correlation coefficient does indeed differ from zero.

Answers: **(i)** 0.499, 1.07; **(ii)** –0.468.

Question 9

The expected value 10.125 is found by integrating f(t) between 2 and 2.5, and multiplying the result by the number of observations, namely 100, as most candidates explained. Subtracting the sum of all the other expected values from 100, as a very few did, is not of course acceptable. The goodness of fit test is carried out by first stating the null and alternative hypotheses, and then comparing the calculated value 5.67 of χ^2 with the tabular value 6.25, concluding that the distribution fits the data. While most candidates performed the test correctly, a few mistakenly combined the last two entries in each of the tables of observed and expected values even though both the expected values concerned are greater than 5.

Question 10

While many candidates found the value ln 2 of k by equating 2^x and e^{kx}, rather fewer were able to find a by equating to unity the integral of f(x) over the infinite range $x > 0$. Simply comparing ae^{-kx} with the usual form of f(x) for a negative exponential distribution and deducing that a must equal k is invalid, since this is to assume what the question requires to be shown. Most candidates were able to write down the value of E(X), though some found it by integrating xf(x). The distribution function G(y) of Y is first shown to equal the distribution function F(x) of X with $x = k^{-1} \ln y$, and this F(x) is then obtained by integration of f(x) with the appropriate limits. Many of those who derived G(y) correctly did not appreciate that their expression was capable of simplification, and similarly for the probability density function g(y) which is obtained by differentiating G(y).

Answers: ln 2; $\frac{1}{\ln 2}$; $1 - \frac{1}{y}$; $\frac{1}{y^2}$ $(y > 1)$.

Question 11 EITHER

This alternative was unpopular, and very few reasonable attempts were seen. The starting point is to observe or deduce that the vertical reaction R_B between the wall and the ladder at B will be a maximum when the dog is at B, perhaps by taking moments for the ladder about A. This equation then yields the corresponding maximum value W of R_B, which resolution of horizontal forces for the ladder shows to be equal to the frictional force F_A at A. The reaction there, $\frac{5W}{4}$, follows from a resolution of the vertical forces on the ladder, and the usual relation $F_A \leqslant \mu R_A$ produces the given inequality for μ. The friction and reaction between the cube and the floor are found by horizontal and vertical resolution of forces to be $F_{cube} = R_B$ and $R_{cube} = (\frac{5}{4} + k)W$. Thus $\mu R_{cube} \geqslant (1 + \frac{4k}{5})W \geqslant W$ while $F_{cube} \leqslant W$ and hence the cube does not slide whatever the value of k. The required smallest value k_{min} of k follows from noting that if the cube were to turn about D then the reaction of the floor would act there and so exert no moment about D. Requiring that the total clockwise moments about D of the other three forces acting on the cube should be non-negative and using $F_A \leqslant W$ yields k_{min}.

Answer: $\frac{11}{8}$.

Question 11 OR

Most candidates stated the null and alternative hypotheses, and produced biased or unbiased estimates of the variances for the populations of bottles filled by the two machines. In the case of unbiased estimates 0.168 and 0.0784, they should be divided by 50 and 60 respectively, and the results added to give an unbiased estimate s^2 of the variance for the combined populations. The corresponding divisors for the biased estimates are 49 and 59. A value of z is then calculated by dividing the difference 0.22 of the sample means by s, yielding 3.22. Comparison with the value 2.054, obtained by interpolation in the main table for the normal distribution function, leads to the conclusion that μ_2 is greater than μ_1. A very common error, however, was to assume that the two populations share the same variance, which seems unreasonable in view of the very different estimates of the individual population variances given above, and therefore to calculate a two-sample estimate 0.119 of this common variance, using the method given in the List of Formulae. Continuation of this approach leads to an estimate 3.33 of z. In the second part of the question another value 1.756 of z is calculated from $\frac{0.22 - 0.1}{s}$, and the normal distribution function table is used to find $\Phi(z) = 0.9604$ and hence the required set of values of α.

Answer: $\alpha \geqslant 4$.