CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2013 series

9231 FURTHER MATHEMATICS

9231/11 Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2013 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a ‘fortuitous' answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

Qu No	Commentary	Solution	Marks	Part Mark	Total
1	Use of $\frac{1}{2}\int r^2\, d\theta$	$A = \frac{1}{2}\int_0^{2\pi} 4(1+2\cos\theta+\cos^2\theta)\, d\theta$	M1		
	Use of double angle formula and attempt to integrate.	$= \int_0^{2\pi}(3+4\cos\theta+\cos 2\theta)\, d\theta$	M1		
	Integrates correctly.	$= \left[3\theta+4\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{2\pi}$	A1		
	Finds value.	$= 6\pi$ (CWO) Accept 18.8	A1	4	[4]
2	Proves base case.	P_n: $5^{2n}-1$ is divisible by 8. $5^2-1=24=3\times 8 \Rightarrow P_1$ is true	B1 B1		
	States inductive hypothesis.	Assume P_k is true: $5^{2k}-1=8\lambda$ for some k. $5^{2k+2}-1=25.5^{2k}-1=24.5^{2k}+5^{2k}-1$ $=3\times 8.5^{2k}+8\lambda$	M1		
	Proves inductive step.	$\therefore P_k \Rightarrow P_{k+1}$	A1		
	States conclusion.	(Since P_1 is true and $P_k \Rightarrow P_{k+1}$) $\therefore$ P_n is **for every positive integer** n (by PMI).	A1	5	[5]
3	Uses $\sum\alpha = \frac{-b}{a}$.	$c = 2$	B1	1	
	Uses substitution	$(\alpha+\beta=c-\gamma$ etc.$) \Rightarrow y=c-x \Rightarrow x=c-y$ $(2-y)^3-2(2-y)^2-3(2-y)+4=0\ldots$(their c)	M1 M1		
	to obtain required cubic equation.	$\Rightarrow y^3-4y^2+y+2=0$	A1	3	
	Obtains equation whose roots are reciprocals of those in previous cubic equation.	Uses $z=y^{-1}$ to obtain $2z^3+z^2-4z+1=0$	M1A1	2	
	Uses $\sum\alpha^2 = \left(\sum\alpha\right)^2 - 2\sum\alpha\beta$	$\sum\frac{1}{(\alpha+\beta)^2} = \left(\frac{1}{2}\right)^2 - 2(-2) = 4\frac{1}{4}$	M1A1	2	[8]

4	Integrates by parts.	$I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx$ $= \left[\frac{x}{(1+x^2)^n}\right]_0^1 + \int_0^1 n(1+x^2)^{-(n+1)}.2x^2\,dx$	M1A1		
	Rearranges.	$= 2^{-n} + 2n\int_0^1 (1+x^2)^{-(n+1)}(1+x^2-1)\,dx$	M1A1		
	Obtains result.	$\therefore 2nI_{n+1} = 2^{-n} + (2n-1)I_n$. (AG)	A1	5	
	Uses reduction formula to find I_2.	$2I_2 = \frac{1}{2} + \frac{1}{4}\pi \Rightarrow I_2 = \frac{1}{4} + \frac{1}{8}\pi$	M1A1		
	Uses reduction formula to find I_3.	$4I_3 = \frac{1}{4} + \frac{3}{4} + \frac{3}{8}\pi \Rightarrow I_3 = \frac{1}{4} + \frac{3}{32}\pi$	A1	3	**[8]**
5	Finds partial fractions.	$\frac{1}{(`2r+1)(2r+3)} = \frac{1}{2}\left\{\frac{1}{2r+1} - \frac{1}{2r+3}\right\}$ $\sum_{r=1}^{N} \frac{1}{(2r+1)(2r+3)}$	M1A1		
	Expresses terms as differences.	$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \frac{1}{2}\left(\frac{1}{2N+1} - \frac{1}{2N+3}\right)$	M1A1		
	Shows cancellation.	$= \frac{1}{6} - \frac{1}{2(2N+3)}$ (AG)	A1	5	
	Uses $\sum_{N+1}^{2N} = \sum_{1}^{2N} - \sum_{1}^{N}$.	$\sum_{N+1}^{2N} = \left(\frac{1}{6} - \frac{1}{2(4N+3)}\right) - \left(\frac{1}{6} - \frac{1}{2(2N+3)}\right)$	M1		
	Applies result	$= \frac{1}{2}\left(\frac{1}{2N+3} - \frac{1}{4N+3}\right)$	A1		
	and simplifies.	$= \frac{N}{(2N+3)(4N+3)}$	M1		
	Deduces inequality.	$< \frac{N}{2N.4N} = \frac{1}{8N}$ (AG)	A1	4	**[9]**

6	Shows **e** is an eigenvector of **A** and gives eigenvalue.	**Ae** = 2**e** $\Rightarrow$ **e** is an eigenvector with eigenvalue 2.	M1A1	2	
	Finds characteristic equation.	$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$	M1A1		
	Factorises.	$\Rightarrow (\lambda - 2)(\lambda^2 - 1) = 0$	A1		
	States other eigenvalues.	Other eigenvalues are –1 and 1.	A1	4	
	Repeats for **B**.	**Be** = 3**e** $\Rightarrow$ **e** is an eigenvector with eigenvalue 3	B1		
	States result for **AB**.	**ABe = A**.3**e** = 3**Ae** = 3.2**e** = 6**e** **AB** has eigenvector **e** with eigenvalue 6	M1A1	3	**[9]**
7	Expands and groups. Use of $z - z^{-1}$ and $z + z^{-1}$. Correctly.	$\left(z - z^{-1}\right)^6 = \left(z^6 + z^{-6}\right) - 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) - 20$	M1A1 M1		
	Obtains result.	$(2\mathrm{i}\sin\theta)^6 = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$ $\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$ (Allow $p = 10$, $q = -15$, $r = 6$, $s = -1$)	A1A1 A1	6	
	Integrates correctly.	$\left[\frac{5\theta}{16} - \frac{15\sin 2\theta}{64} + \frac{3\sin 4\theta}{64} - \frac{\sin 6\theta}{192}\right]_0^{\frac{\pi}{4}}$	M1A1		
	Inserts limits and evaluates.	$\frac{5\pi}{64} - \frac{15}{64} + \frac{1}{192} = \frac{5\pi}{64} - \frac{11}{48}$ (SC: 1f power of 2 consistently wrong ¾ for 2nd part.)	M1A1	4	**[10]**

8	Reduces matrices to echelon form.	$\begin{pmatrix} 1 & -2 & 3 & 5 \\ 3 & -4 & 17 & 33 \\ 5 & -9 & 20 & 36 \\ 4 & -7 & 16 & 29 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & -2 & 3 & 5 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & -2 & 0 & -3 \\ 2 & -1 & 0 & 0 \\ 4 & -7 & 1 & -9 \\ 6 & -10 & 0 & -14 \end{pmatrix} \to \ldots \to \begin{pmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$	M1 A1 A1		
	Finds basis for each null space.	$\left.\begin{array}{r} x-2y+3z+5t=0 \\ y+4z+9t=0 \\ z+2t=0 \end{array}\right\} \Rightarrow \left\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}\right\}$ $\left.\begin{array}{r} x-2y-3t=0 \\ y+2t=0 \\ z+t=0 \end{array}\right\} \Rightarrow \left\{\begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}\right\}$	M1A1 A1	6	
	Writes $\mathbf{x}_1$ and $\mathbf{x}_2$ appropriately.	$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}$	B1√		
	Finds difference and solves.	$\Rightarrow \mathbf{x}_1 - \mathbf{x}_2 = \begin{pmatrix} \lambda-\mu \\ \lambda-2\mu \\ 2\lambda-\mu \\ -\lambda+\mu \end{pmatrix} \Rightarrow \lambda - 2\mu = 5$ and $2\lambda - \mu = 7$ $\Rightarrow \lambda = 3$ and $\mu = -1$ $\mathrm{x}_1 - \mathrm{x}_2 = (4 \;\; 5 \;\; 7 \;\; -4)^{\mathrm{T}} \Rightarrow p = 4$ and $q = -4$	M1 A1 A1	4	**[10]**

9	Finds complementary function.	$4m^2+4m+1=0 \Rightarrow (2m+1)^2=0 \Rightarrow m=-\frac{1}{2}$	M1		
		C.F.: $x = A\mathrm{e}^{-\frac{t}{2}} + Bt\mathrm{e}^{-\frac{t}{2}}$	A1		
	Finds particular integral.	P.I.: $x = k\mathrm{e}^{-2t} \Rightarrow \dot{x} = -2k\mathrm{e}^{-2t} \Rightarrow \ddot{x} = 4k\mathrm{e}^{-2t}$	M1		
		$16k-8k+k=6 \Rightarrow k=\frac{2}{3} \Rightarrow x=\frac{2}{3}\mathrm{e}^{-2t}$	A1		
	Adds for general solution.	G.S.: $x = A\mathrm{e}^{-\frac{t}{2}} + Bt\mathrm{e}^{-\frac{t}{2}} + \frac{2}{3}\mathrm{e}^{-2t}$	A1		
	Uses initial conditions to find constants.	$x(0)=\frac{5}{3} \Rightarrow \frac{5}{3} = A+\frac{2}{3} \Rightarrow A=1$	B1		
		$\dot{x} = -\frac{1}{2}\mathrm{e}^{-\frac{t}{2}} + B\mathrm{e}^{-\frac{t}{2}} - \frac{1}{2}Bt\mathrm{e}^{-\frac{t}{2}} - \frac{4}{3}\mathrm{e}^{-2t}$	M1		
		$\dot{x}(0)=\frac{7}{6} \Rightarrow \frac{7}{6} = -\frac{1}{2}+B-\frac{4}{3} \Rightarrow B=3$	A1		
	Gives particular solution.	$x = \mathrm{e}^{-\frac{t}{2}} + 3t\mathrm{e}^{-\frac{t}{2}} + \frac{2}{3}\mathrm{e}^{-2t}$	A1	9	
	States limit.	$\lim_{t\to\infty} x = 0$	B1	1	[10]

10	States asymptotes.	Vertical: $x=1$ and Horizontal: $y=2$	B1B1	2	
	Obtains quadratic form in x.	$yx^2-2yx+y=2x^2-3x-2$ $\Rightarrow (y-2)x^2-(2y-3)x+(y+2)=0$	M1A1		
	Uses $B^2-4AC\geq 0$ for real roots.	For real x $(2y-3)^2-4(y-2)(y+2)\geq 0$ $\Rightarrow 12y\leq 25 \Rightarrow y\leq \frac{25}{12}$.	M1 A1	4	
	Finds condition for $y'=0$.	$y'=0\Rightarrow$ $(x^2-2x+1)(4x-3)-(2x^2-3x-2)(2x-2)=0$	M1		
	Solves	$\Rightarrow x^2-8x+7=0\Rightarrow (x-7)(x-1)=0$ $\Rightarrow x=7$, (since $x=1$ is vertical asymptote).	A1		
	Obtains stationary point.	Stationary point is $\left(7,\frac{25}{12}\right)$	A1	3	
	Sketch showing:	Axes and asymptotes (-0.5,0) , (2,0) , (0,–2) and (4,2) Left hand branch. Right hand branch.	B1 B1 B1 B1	4	**[13]**

11 E	Differentiation.	$y = 2\sec x \Rightarrow y' = 2\sec x \tan x$	M1		
	Use of $\sec^2 x = 1 + \tan^2 x$.	$1 + (y')^2 = 1 + 4\sec^2 x(\sec^2 x - 1)$ $= 4\sec^4 x - 4\sec^2 x + 1$ $= (2\sec^2 x - 1)^2$	M1 A1		
	Substitute in arc length formula.	$s = \int_0^{\frac{\pi}{4}} (2\sec^2 x - 1)\,dx$	A1	4	
	Integrate.	$= [2\tan x - x]_0^{\frac{\pi}{4}}$	M1		
	Substitute limits.	$= \left[2 - \frac{1}{4}\pi\right]$	A1	2	
(i)	Use surface area formula	$S = 2\pi \int_0^{\frac{\pi}{4}} 2\sec x(2\sec^2 x - 1)\,dx$	M1A1		
	Obtain correct form.	$= 4\pi \int_0^{\frac{\pi}{4}} (2\sec^3 x - \sec x)\,dx$ (AG)	A1	3	
(ii)	Differentiates	$\frac{d}{dx}(\sec x \tan x) = \sec x \tan^2 x + \sec^3 x$ $= \sec x(\sec^2 x - 1) + \sec^3 x$	M1A1		
	and obtains printed result.	$= 2\sec^3 x - \sec x$ (AG)	A1	3	
	Uses result to deduce surface area.	$S = 4\pi[\sec x \tan x]_0^{\frac{\pi}{4}}$ $= 4\pi\sqrt{2}$	M1 A1	2	**[14]**

11 O	Vector perpendicular to Π_1	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 1 \\ 1 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix}$	M1A1		
	Obtains cartesian equation	$3 \times 2 + 1 \times (-1) + (-2) \times (-6) = 17$ $\Rightarrow 3x - y - 6z = 17$	M1 A1	4	
	Obtains area of triangle *ABC*.	$\frac{1}{2}\sqrt{3^2 + 1^2 + 6^2} = \frac{1}{2}\sqrt{46}$ (=3.39)	M1 A1		
	Obtains length of perpendicular from *D* to triangle *ABC*.	$\left\lvert \frac{9 - 6 - 12 - 17}{\sqrt{3^2 + 1^2 + 6^2}} \right\rvert = \frac{26}{\sqrt{46}}$	M1A1		
	Uses $\frac{1}{3} \times$ Base area $\times$ Height.	Either $\frac{1}{3} \times \frac{1}{2}\sqrt{46} \times \frac{26}{\sqrt{46}} = \frac{13}{3}$	M1A1		
	or triple scalar product method.	Or e.g. $\left\lvert \frac{1}{6}\begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix} \right\rvert = \frac{26}{6} = \frac{13}{3}$		6	
	Obtains normal to *ABD*.	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 5 & 4 \\ 2 & 0 & 1 \end{vmatrix} = \begin{pmatrix} 5 \\ 7 \\ -10 \end{pmatrix}$	M1A1		
	Uses scalar product	$\sqrt{3^2 + 1^2 + 6^2}\sqrt{5^2 + 7^2 + 10^2}\cos\theta = \begin{pmatrix} 3 \\ -1 \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 7 \\ -10 \end{pmatrix}$	M1		
	to find angle between normals and hence angle between Π_1 and Π_2.	$\Rightarrow \cos\theta = \frac{68}{\sqrt{46}\sqrt{174}} \Rightarrow \theta = 40.5°$	A1	4	**[14]**