

MARK SCHEME for the May/June 2012 question paper
for the guidance of teachers

9231 FURTHER MATHEMATICS

9231/13

Paper 1, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes must be read in conjunction with the question papers and the report on the examination.

- Cambridge will not enter into discussions or correspondence in connection with these mark schemes.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.
- The symbol $\checkmark$ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.
- Note: B2 or A2 means that the candidate can earn 2 or 0.
B2/1/0 means that the candidate can earn anything from 0 to 2.

The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.
- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking g equal to 9.8 or 9.81 instead of 10.

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The following abbreviations may be used in a mark scheme or used on the scripts:

AEF	Any Equivalent Form (of answer is equally acceptable)
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
BOD	Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)
CAO	Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)
CWO	Correct Working Only – often written by a 'fortuitous' answer
ISW	Ignore Subsequent Working
MR	Misread
PA	Premature Approximation (resulting in basically correct work that is insufficiently accurate)
SOS	See Other Solution (the candidate makes a better attempt at the same question)
SR	Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1	A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through $\sqrt{}$ " marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.
PA –1	This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

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Qu No	Commentary	Solution	Marks	Part Mark	Total
1	Finds partial fractions. Use method of differences. Obtains results.	$\frac{1}{r(r+2)} = \frac{1}{2} \left\{ \frac{1}{r} - \frac{1}{r+2} \right\}$ $\sum_{r=1}^n \frac{1}{r(r+2)} =$ $\frac{1}{2} \left\{ \left[\frac{1}{n} - \frac{1}{n+2} \right] + \left[\frac{1}{n-1} - \frac{1}{n+1} \right] + \dots + \left[\frac{1}{2} - \frac{1}{4} \right] + \left[1 - \frac{1}{3} \right] \right\}$ $= \frac{1}{2} \left\{ \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right\} \text{ (acf)} \Rightarrow S_{\infty} = \frac{3}{4}$	M1A1 M1 A1A1√		
2	(States proposition.) Proves base case. States Inductive hypothesis. Proves inductive step. States conclusion.	$(P_n : u_n = 4 \left(\frac{3}{4} \right)^n - 2)$ Let $n = 1$ $4 \times \frac{3}{4} - 2 = 3 - 2 = 1 \Rightarrow P_1$ true. Assume P_k is true for some k. $u_{k+1} = \frac{3 \left\{ 4 \left(\frac{3}{4} \right)^k - 2 \right\} - 2}{4} = 4 \cdot \frac{3}{4} \cdot \left(\frac{3}{4} \right)^k - \frac{6+2}{4}$ $= 4 \cdot \left(\frac{3}{4} \right)^{k+1} - 2 \quad \therefore P_k \Rightarrow P_{k+1}$ $\therefore$ By PMI P_n is true $\forall$ positive integers.	B1 B1 M1 A1 A1		[5]
3		$y + x \frac{dy}{dx} + 3(x+y)^2 \left(1 + \frac{dy}{dx} \right) = 0$ $0 + y' + 3 + 3y' = 0$ $\Rightarrow y' = -\frac{3}{4} \quad \text{(AG)}$ $\frac{dy}{dx} + \frac{dy}{dx} + x \frac{d^2y}{dx^2} + 6(x+y) \left(1 + \frac{dy}{dx} \right)^2 + 3(x+y)^2 \frac{d^2y}{dx^2} = 0$ $-\frac{3}{4} - \frac{3}{4} + y'' + 6 \times \frac{1}{16} + 3y'' = 0$ $\Rightarrow y'' = \frac{9}{32}$ N.B. Mark similarly if expression expanded before differentiating.	B1B1 B1 B1 B1B1 M1 A1	3 5	[8]

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Qu No	Commentary	Solution	Marks	Part Mark	Total
4	Integrates by parts. Obtains reduction formula. Finds I_0 (or I_1) and uses reduction formula. (M1A1 for I_1 if found immediately.) Obtains I_2 . Obtains I_3 .	$I_n = \int_1^e x^2 (\ln x)^n dx$ $= \left[(\ln x)^n \frac{x^3}{3} \right]_1^e - \int_1^e n(\ln x)^{n-1} \cdot \frac{1}{x} \cdot \frac{x^3}{3} dx$ $= \frac{e^3}{3} - \frac{n}{3} I_{n-1} \quad (\text{AG})$ $I_0 = \int_1^e x^2 dx = \left[\frac{x^3}{3} \right]_1^e = \frac{e^3 - 1}{3}$ $\Rightarrow I_1 = \frac{e^3}{3} - \frac{1}{3} \left(\frac{e^3 - 1}{3} \right) = \frac{2e^3 + 1}{9}$ $\Rightarrow I_2 = \frac{e^3}{3} - \frac{2}{3} \left(\frac{2e^3 + 1}{9} \right) = \frac{5e^3 - 2}{27}$ $\Rightarrow I_3 = \frac{e^3}{3} - \left(\frac{5e^3 - 2}{27} \right) = \frac{4e^3 + 2}{27}$	M1A1 A1 A1 B1 M1 A1 A1	4 4	[8]
5	Proves initial result. Finds eigenvectors corresponding to given eigenvalues. Finds corresponding eigenvalue. Recognises the result proved initially. Gives eigenvalues and matches eigenvectors.	$(\mathbf{A} + k\mathbf{I})\mathbf{e} = \mathbf{A}\mathbf{e} + k\mathbf{I}\mathbf{e} = \lambda\mathbf{e} + k\mathbf{e} = (\lambda + k)\mathbf{e}$ $\therefore (\mathbf{A} + k\mathbf{I})$ has eigenvalue $(\lambda + k)$ with corresponding eigenvector $\mathbf{e}$. Eigenvalues are -3 and 4 (given). Corresponding eigenvectors are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ Third eigenvalue is 6. $\mathbf{C} = \mathbf{B} - 3\mathbf{I}$ (Stated or implied.) Eigenvalues: $-6, 1, 3$. Corresponding eigenvectors are: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$. (OE) (For 'non hence' method, using characteristic equation, award B1 rather than M1A1 for eigenvalues, followed by B1 for eigenvectors.)	M1A1 M1A1 A1 B1 M1 A1 A1√	2 3 1 3	[2] [3] [1] [3]

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Qu No	Commentary	Solution	Marks	Part Mark	Total
8 (i)	Deduces initial result. Substitutes into cubic equation. Deduces new cubic equation.	$u = -\alpha + \beta + \gamma \Rightarrow u + 2\alpha = \alpha + \beta + \gamma = 1$ $\Rightarrow \alpha = \left(\frac{1-u}{2}\right)$ $\Rightarrow \left(\frac{1-u}{2}\right)^3 - \left(\frac{1-u}{2}\right)^2 - 3\left(\frac{1-u}{2}\right) - 10 = 0$ $\Rightarrow \dots \Rightarrow u^3 - u^2 - 13u + 93 = 0$	M1A1 M1 A1 A1	5	
(ii)	Deduces initial result. Substitutes into cubic equation. Deduces new cubic equation.	$\alpha\beta\gamma = 10$ $\Rightarrow v = \frac{1}{\beta\gamma} \Rightarrow \frac{v}{\alpha} = \frac{1}{\alpha\beta\gamma} = \frac{1}{10} \Rightarrow \alpha = 10v$ $(10v)^3 - (10v)^2 - 3(10v) - 10 = 0$ $\Rightarrow 100v^3 - 10v^2 - 3v - 1 = 0$	B1 M1A1 M1 A1	5	[10]
8 (i)	Alternatively: For final 3 marks in (i): Award M1 for an attempt at formulae for all three coefficients. A1 for any two correct. A1 for completion	Let equation be $u^3 + bu^2 + cu + d = 0$. $-b = \sum \alpha = 1 \Rightarrow b = -1$ $c = 4\sum \alpha\beta - (\sum \alpha)^2$ $= 4 \times (-3) - 1^2 = -13$ $-d = 4\sum \alpha \sum \alpha\beta - (\sum \alpha)^3 - 8\alpha\beta\gamma$ $= 4 \times 1 \times (-3) - 1^3 - 8 \times 10 = -93$ So $u^3 - u^2 - 13u + 93 = 0$		(5)	
(ii)	For final 4 marks in (ii): Award M1 for an attempt at formulae for all three coefficients. A1 for any one correct. A1 for a second one correct. A1 for completion.	Let equation be $v^3 + bv^2 + cv + d = 0$. $-b = \frac{\sum \alpha}{\alpha\beta\gamma} = \frac{1}{10} \Rightarrow b = -\frac{1}{10}$ $c = \frac{\sum \alpha\beta}{(\alpha\beta\gamma)^2} = \frac{-3}{10^2} = -\frac{3}{100}$ $-d = \frac{1}{(\alpha\beta\gamma)^2} = \frac{1}{10^2} \Rightarrow d = -\frac{1}{100}$ So $v^3 - \frac{1}{10}v^2 - \frac{3}{100}v - \frac{1}{100} = 0$ or $100v^3 - 10v^2 - 3v - 1 = 0$.		(5)	[10]

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Qu No	Commentary	Solution	Marks	Part Mark	Total
9	Finds normal vector to plane. Uses known point to find constant term. Angle between normals is equal to angle between planes. Solve plane equation simultaneously. (Note: may find direction from vector product and use with one point.)	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -1 \\ 1 & 2 & -2 \end{vmatrix} = \begin{pmatrix} 6 \\ 1 \\ 4 \end{pmatrix} \quad \text{or} \quad \begin{cases} x = 2 + \lambda + \mu \\ y = -3 - 2\lambda + 2\mu \\ z = 1 - \lambda - 2\mu \end{cases}$ Plane equation is $6x + y + 4z = \text{constant}$. Substitute $(2, -3, 1) \Rightarrow 12 - 3 + 4 = 13$. Or eliminate λ and μ. $\Rightarrow \Pi_1: 6x + y + 4z = 13$ $\cos \theta = \frac{(6\mathbf{i} + \mathbf{j} + 4\mathbf{k}) \cdot (3\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})}{\sqrt{6^2 + 1^2 + 4^2} \sqrt{3^2 + 2^2 + 3^2}} = \frac{4}{\sqrt{53}\sqrt{22}}$ $\Rightarrow \theta = 83.3^\circ \quad \text{or} \quad 1.45 \text{ rad.}$ $6x + y + 4z = 13$ and $3x - 2y - 3z = 4$ Obtains e.g. $y + 2z = 1$ and $3x + z = 6$ Or two of $(0, -11, 6)$, $(11/6, 0, 1/2)$, $(2, 1, 0)$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -6 \\ 3 \end{pmatrix} \quad (\text{OE})$ Alternatively: Direction of line from vector product. Finds a point on line. States equation of line.	M1A1 M1 A1 M1A1 A1 M1 A1A1 A1 (M1A1) (A1) (A1)	4 3 4	[11]

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Qu No	Commentary	Solution	Marks	Part Mark	Total
10	Reduces augmented matrix	$\begin{pmatrix} 1 & -2 & -2 & -7 \\ 2 & a-9 & -10 & -11 \\ 3 & -6 & 2a & -29 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & -2 & -7 \\ 0 & a-5 & -6 & 3 \\ 0 & 0 & 2a+6 & -8 \end{pmatrix}$	M1A1		
	Obtains set of values for a , giving unique solutions.	Unique solution for all real a except $a = -3$ or 5 Alternatively for first two marks: $\begin{vmatrix} 1 & -2 & -2 \\ 2 & (a-9) & -10 \\ 3 & -6 & 2a \end{vmatrix} \neq 0 \Rightarrow (a-5)(a+3) \neq 0$	A1A1	4	
	(i) Case of no solutions.	$a = -3 : 2a + 6 = 0$ $\Rightarrow 0z = -8 \Rightarrow$ no solutions (AG)	M1A1✓	2	
	(ii) Case of infinite solutions.	$a = 5 : \Rightarrow z = -\frac{1}{2}$ and $x - 2y = -8$ (*) $\therefore$ infinite number of solutions (AG)	M1A1✓	2	
	Obtains particular solution.	$z = -\frac{1}{2}$ and $x + y + z = 2 \Rightarrow x + y = \frac{5}{2}$ Solving simultaneously with (*) gives $x = -1 \quad y = \frac{7}{2} \quad z = -\frac{1}{2}$	B1 M1A1	3	[11]

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Qu No	Commentary	Solution	Marks	Part Mark	Total
11	EITHER Uses $x^2 + y^2 = r^2$, $x = r \cos \theta$ and $y = r \sin \theta$. One mark for each loop, or half of whole curve. Uses sector area formula. S.C. Omission of $\frac{1}{2}$ factor, but correct integration gets B1. Differentiates. Puts $y' = 0$. Obtains coordinates.	$(x^2 + y^2)^2 = a^2(x^2 - y^2)$ $\Rightarrow r^2 = a^2 \left(\frac{x^2}{r^2} - \frac{y^2}{r^2} \right)$ $= a^2 (\cos^2 \theta - \sin^2 \theta) = a^2 \cos 2\theta \text{ (AG)}$ Sketches C. $\text{Area} = \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta = \int_0^{\frac{\pi}{4}} a^2 \cos 2\theta \, d\theta$ $= a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{4}} = \frac{a^2}{2}$ $2(x^2 + y^2)(2x + 2yy') = a^2(2x - 2yy')$ $y' = 0 \Rightarrow 2x(x^2 + y^2) = a^2 x$ $\Rightarrow 2r^2 = a^2 \Rightarrow r = \frac{a}{\sqrt{2}} \quad (r \geq 0)$ $\Rightarrow \cos 2\theta = \frac{1}{2}$ $\Rightarrow \theta = \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}$ i.e. $\left(\frac{a}{\sqrt{2}}, \pm \frac{\pi}{6} \right)$ and $\left(\frac{a}{\sqrt{2}}, \pm \frac{5\pi}{6} \right)$	M1 A1 B2,1,0 M1 A1A1 B1B1 M1 A1 M1 A1A1	2 2 3 7	[14]

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Qu No	Commentary	Solution	Marks	Part Mark	Total
	Alternatively for last 7 marks: Obtains condition for tangent parallel to initial line. Differentiates equation of C. Forms equation. Solves for $\tan\theta$, and θ. Writes coordinates of points.	$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{d\theta} = 0 \Rightarrow r \cos \theta + \frac{dr}{d\theta} \sin \theta = 0$ $2r \frac{dr}{d\theta} = -2a^2 \sin 2\theta \Rightarrow \frac{dr}{d\theta} = -\frac{a^2}{r} \sin 2\theta$ $\therefore r \cos \theta - \frac{a^2}{r} \sin 2\theta \sin \theta = 0$ $\therefore a^2 \cos 2\theta \cos \theta = a^2 \sin 2\theta \sin \theta$ $\therefore \frac{1}{\tan 2\theta} = \tan \theta$ $\therefore \frac{1-t^2}{2t} = t \Rightarrow 2t^2 = 1-t^2 \Rightarrow 3t^2 = 1$ $\therefore t = \pm \frac{1}{\sqrt{3}}$ $\therefore \theta = \pm \frac{\pi}{6}, \pm \frac{5\pi}{6}$ $\left(\frac{a}{\sqrt{2}}, \pm \frac{\pi}{6} \right), \left(\frac{a}{\sqrt{2}}, \pm \frac{5\pi}{6} \right)$ (Award final A1 if r found but result not written out.)	(M1A1) (A1) (M1) (M1) (A1) (A1)	(7)	[14]

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11	OR				
	Differentiates once.	$\frac{dy}{dx} = z + x \frac{dz}{dx}$	B1		
	Differentiates again.	$\frac{d^2 y}{dx^2} = 2 \frac{dz}{dx} + x \frac{d^2 z}{dx^2}$	B1		
	Substitutes.	$\frac{d^2 z}{dx^2} + \left(\frac{2}{x} + 6 - \frac{2}{x} \right) \frac{dz}{dx} + \left(\frac{6z}{x} - \frac{2z}{x^2} + 9z - \frac{6z}{x} + \frac{2z}{x^2} \right) = 169 \sin 2x$	M1		
	Obtains result	$\Rightarrow \frac{d^2 z}{dx^2} + 6 \frac{dz}{dx} + 9z = 169 \sin 2x \quad (\text{AG})$ (Mark similarly if substitution is rearranged to $z = \frac{y}{x}$.)	A1	4	
	Finds CF.	$m^2 + 6m + 9 = (m + 3)^2 = 0 \Rightarrow m = -3$ CF: $Ae^{-3x} + Bxe^{-3x}$	M1 A1		
	Finds PI.	PI: $y = p \sin 2x + q \cos 2x$ $y' = 2p \cos 2x - 2q \sin 2x$ $y'' = -4p \sin 2x - 4q \cos 2x$ $5p - 12q = 169$ $12p + 5q = 0$ $\Rightarrow p = 5 \text{ and } q = -12$	M1 A1		
	Finds GS.	GS: $z = Ae^{-3x} + Bxe^{-3x} + 5 \sin 2x - 12 \cos 2x$	A1		
	Evaluates coefficients from initial conditions.	$-10 = A - 12 \Rightarrow A = 2$ $z' = -6e^{-3x} + Be^{-3x} - 3Bxe^{-3x} + 10 \cos 2x + 24 \sin 2x$ $5 = -6 + B + 10 \Rightarrow B = 1$	B1 M1 A1		
	Finds particular solution.	$z = 2e^{-3x} + xe^{-3x} + 5 \sin 2x - 12 \cos 2x$ $\therefore y = 2xe^{-3x} + x^2 e^{-3x} + 5x \sin 2x - 12x \cos 2x$	A1	10	[14]