FURTHER MATHEMATICS

Paper 9231/01
Paper 1

General comments

Some scripts of high quality and many of a good quality were received in response to this examination. There were very few poor scripts. Work was generally well presented by all, except the weakest candidates. Solutions were set out in a clear logical order and the standard of algebra and numerical accuracy was good.

There was no evidence to suggest that candidates had any difficulty completing the paper in the time allowed. The vast majority of scripts had substantial attempts at all questions. Once again there were very few misreads and this year there were few rubric infringements.

The Examiners generally felt that candidates had a sound knowledge of most topics on the syllabus. Induction and linear spaces still remain as areas of uncertainty, while improvement seems to have been made in curve sketching and vector work.

Comments on specific questions

Question 1

There were many complete and accurate answers to this question. Almost all candidates were able to establish the first result

$$\frac{1}{n^2+1}-\frac{1}{(n+1)^2+1}=\frac{2n+1}{(n^2+1)(n^2+2n+2)}.$$

Most were then able to use the method of difference to find

$$S_N=\frac{1}{2}-\frac{1}{(N+1)^2+1}.$$

It was then necessary to explain that since $(N+1)^2$ was positive for $N > 1$, then $\frac{1}{(N+1)^2+1}>0$ and $S_N<\frac{1}{2}$. Many candidates lost the mark for this piece of work, but, nevertheless, were able to state correctly that S_∞ was $\frac{1}{2}$.

Answer: $\frac{1}{2}$.

Question 2

Most candidates were able to differentiate correctly the given expressions for x and y with respect to t and then show that $\left(\frac{dx}{dt}\right)^2=\left(\frac{dy}{dt}\right)^2=\left(1+\frac{1}{t}\right)^2$.

Many knew the correct integral representation for the area of the surface generated and were able to successfully obtain the printed result.

Question 3

The vast majority of candidates produced a completely correct solution to this question. Almost all candidates knew how to derive the complementary function. There were only a few slips with the algebra involved and in a small number of cases the complementary function appeared as $e^{-2x}(A\cos 5\theta + B\sin 5\theta)$.

Almost all candidates knew how to obtain $2x+1$ as the particular integral and hence find the general solution correctly.

Answer: $y = e^{-2x}(A\cos 5x + B\sin 5x) + 2x + 1$

Question 4

The key to this question was the realisation that the given equation had to be differentiated implicitly with respect to x. A minority of candidates did not realise this.

The ability to differentiate correctly was somewhat variable and certainly many attempts, even successful ones, could have been far more concise. The Examiners hoped to see something along the lines of

$$y' = 1 - (y + xy')e^{-xy}$$

$$y'' = -(2y' + xy'')e^{-xy} + (y + xy')^2 e^{-xy}.$$

Sadly few candidates were able to be this concise, and errors occurred, particularly when differentiating for the second time.

It was then required to find $y(0)$, $y'(0)$ and $y''(0)$. Numerically correct answers, with incorrect working, did not gain marks.

Occasionally, successful use was made of logarithms or a helpful substitution.

Answer: 1.

Question 5

Many candidates were unable to start the first part of this question. Here it was necessary to write x as $e^{\theta}\cos\theta$ and, using differentiation, find its maximum value in the range $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Candidates fared much better on the second part of the question and nearly all knew the correct integral representation and were successfully able to find the required area.

Pleasingly, most candidates followed the instruction to give *exact* answers.

Answers: $\frac{1}{\sqrt{2}} e^{\frac{\pi}{4}}$; $\frac{1}{4}\left(e^{\pi} - 1\right)$.

Question 6

This question was done well by many candidates. The eigenvectors for matrix **A** were usually found by solving a set of equations, or more elegantly by using a vector product. A considerable number of candidates abandoned the question at this point. A sizeable number of those candidates, who continued with the question, did not heed the word *hence* in the question and proceeded to find the eigenvalues of matrix **B** by setting up and solving the characteristic equation of matrix **B**. Candidates should have observed that **B** = **A** + 3**I** and hence the eigenvalues of **B** were 3 greater than those of **A**. Most candidates knew that the eigenvectors were unchanged.

Answers: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$; 4, 5, 7; $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

Question 7

The first part of the question caused difficulty for some candidates who did not realise that the $3\sqrt{y}$ term needed to be isolated after the substitution had been made. Those who made this step then cubed both sides of the equation to obtain the required result.

The second part of the question was often done better than the first part. The most frequent method was to use the result

$\sum_3 - 3\sum_2 + 30\sum_1 - 3 = 0$ where $\sum_n = \alpha^{3n} + \beta^{3n} + \gamma^{3n}$.

The values $\sum_1 = 3$ and $\sum_2 = -51$ were then inserted to obtain $\sum_3$.

The alternative was to use

$\sum\alpha^3 = (\sum\alpha)^3 - 3(\sum\alpha)(\sum\beta\gamma) + 3\alpha\beta\gamma$ on the equation in y.

Answer: –240.

Question 8

This proved to be the most difficult question for the candidates and the least well done question on the paper.

Often only two marks were scored for stating the inductive hypothesis $H_k : x_k > \frac{1}{2}$ for some integer $k \geq 1$ and for stating that H_1 is true since $x_1 = 1$ and $1 > \frac{1}{2}$. Some candidates found it difficult to express either, or both, of these succinctly.

The two most successful approaches, thereafter, were either to show $x_{k+1} - \frac{1}{2} = \frac{6\left(x_k - \frac{1}{2}\right)}{10 + 4x_k}$ from which it is easy to deduce that $H_k \Rightarrow H_{k+1}$; or to write $x_{k+1} = 2 - \frac{9}{2x_k + 5}$ and show $\frac{9}{2x_k + 5} < \frac{3}{2}$, hence $H_k \Rightarrow H_{k+1}$.

The second part of the question did not require a proof by induction, although some candidates thought that it did. Observing that $x_n - x_{n+1} = \frac{2x_n^2 + x_n - 1}{5 + 2x_n} = \frac{(2x_n - 1)(x_n + 1)}{(5 + 2x_n)}$ it is easy to show that $x_n - x_{n+1} > 0$ and hence the required result.

Question 9

This question was tackled quite successfully by many candidates. Some were unable to do the necessary algebra in the middle of the first part of the question, but the displayed result enabled them to go on and do well on the rest of the question.

Some had difficulty evaluating I_1, while others evaluated I_1 and I_2, not seeing the need for using the reduction formula. Any approach leading to a correct value to 3 decimal places was acceptable here. This included the use of a graphical calculator, as an *exact* answer had not been specified. The latter, however, was full marks or no marks.

Answer: 0.138.

Question 10

Many candidates were able to do part **(i)** correctly, but those who made errors in calculating the vector product for the direction of line l_3 paid a heavy penalty in this question, as they could only earn method marks in parts **(ii)** and **(iii)**, both of which depended on the direction of l_3.

The most direct method of doing part **(ii)** was to find the vector from S to the given point on l_3 and find the magnitude of the vector product between this vector and the unit vector in the direction of l_3. Curiously, had candidates adopted the unorthodox approach of finding the intersection of l_2 and l_3 in answering part **(i)**, then part **(ii)** would have become very easy, as this point is the foot of the perpendicular from S to l_3.

The direction of the normal to the required plane is found by the vector product of the direction of the line l_3 and the vector from P to the given point on l_3. It then remained to find the magnitude of the scalar product between the vector from S to the given point on l_3 and the unit normal vector to the plane. It was not necessary to calculate the equation of the plane, as many candidates did, nor to start finding distances of a set of parallel planes from the origin, all of which wasted valuable time.

There were a pleasing number of completely correct solutions to this question. Many candidates, however, would have benefited by drawing a clear sketch.

Answers: **(i)** $\mathbf{r} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$; **(ii)** $\sqrt{11}$; **(iii)** $\frac{1}{\sqrt{5}}$.

Question 11

Part **(a)** was done reasonably well by the majority of candidates. They knew that it was necessary to expand, $(\cos\theta + \mathrm{i}\sin\theta)^8$, using the binomial theorem and pick the imaginary part to find $\sin 8\theta$. They also attempted to substitute $(1-\sin^2\theta)$ for $\cos\theta$, which earned them most of the marks. There were, however, quite a few unnecessary slips with signs and algebra.

Many could not get started on part **(b)**, but those who realised that they needed to find the imaginary part of the sum of an appropriate geometric progression made good progress. The most efficient solutions retained the complex numbers in exponential form for as long as possible. Some candidates showed impressive mastery of this area of the syllabus.

Answer: **(a)** -128.

Question 12 EITHER

This was the more popular of the two alternatives. The vast majority were able to obtain the two asymptotes in part **(i)**, but a significant minority thought the oblique asymptote was $y = \lambda x$. Most were able to get y' in a suitable form, but some forgot to explain why it was always positive if $\lambda > 0$.

Marks were often lost in part **(iii)** however. Many were able to show that stationary points, if they existed, were between -4 and 0, but the question asked for more than this. One approach was to solve the quadratic obtained from $y' = 0$ and use the condition $\lambda < -\frac{1}{2}$ to show distinct solutions were both negative. Alternatively, some candidates found the discriminant of the equation obtained from $y' = 0$ and used the condition to show that it was positive, thus implying distinct roots. They then showed both roots were negative since the sum of the roots of the equation was negative, whilst the product of the roots was positive.

Most graphs were along the right lines, but marks were lost for poor forms at infinity, or the right hand branch not passing through the origin or plotting specific examples, rather than being general.

Answers: **(i)** $x = -2$, $y = \lambda x + 1$.

Question 12 OR

Most candidates attempting this alternative were able to reduce the matrix to echelon form and deduce that its rank was 2. Then, using a system of equations from the reduced matrix, they obtained, mostly successfully, the basis for the null space.

Almost every candidate getting this far was able to evaluate the matrix product correctly. The next step, however, caused problems, because many candidates did not appreciate the distinction between showing that the given form was a solution of the equation and what they were asked to show, which was that any solution of the equation had the given form.

There were very few good attempts at the last part of the question. This required setting up two equations, one for A and one for B, both in terms of λ and μ. The equations then had to be solved for λ and μ. Thus the solution could be given in terms of A and B.

Answers : **(i)** 2; **(ii)** $\left[\begin{pmatrix}2\\0\\1\\-1\end{pmatrix},\begin{pmatrix}2\\1\\0\\0\end{pmatrix}\right]$; **(iii)** $\begin{pmatrix}5\\11\\17\\27\end{pmatrix}$; **(iv)** $\begin{pmatrix}A\\B-A-1\\\frac{1}{2}-B+\frac{3A}{2}\\\frac{1}{2}-\frac{3A}{2}+B\end{pmatrix}$.

FURTHER MATHEMATICS

Paper 9231/02
Paper 2

General comments

As has increasingly been the case in recent years, there seemed no great disparity in general performance between Mechanics and Statistics, and in particular this was the case with the final question.

The marks earned by individual candidates varied widely, so in this sense the paper worked well in discriminating between candidates. Even some of those who performed well, though, would benefit from noting that in questions which require the proof of a given answer, sufficient detail should be given to make the method of proof evident to the Examiners. Examples of this are the first part of **Question 2** and the last part of **Question 8**, where simply writing down a single expression without explanation or justification may be insufficiently convincing.

Comments on specific questions

Question 1

Although the majority of candidates introduced two equations relating the velocities of the two particles before and after their collision, selected from the three possibilities of conservation of momentum, the restitution equation and conservation of kinetic energy, a significant proportion had signs for the various terms which were inconsistent between the equations or with the directions of the speeds shown in their diagram. Others failed to relate the initial speeds to the impulse I in deriving their final solutions (the former being $\frac{I}{m_1}$ and $\frac{I}{m_2}$, with the result that many attempts gained only partial marks.

Answers: $\frac{I}{m_1}$, $\frac{I}{m_2}$.

Question 2

As indicated in the general comments above, the Examiners preferred to see the explicit application of the parallel axes theorem to both rods in order to produce the moments of inertia $\frac{4ma^2}{3}$ and $\frac{10ma^2}{3}$ about A, yielding the quoted moment of inertia I for the system. The angular speed ω follows from equating the rotational energy $\frac{1}{2}I\omega^2$ to the change in potential energy. The latter was rarely found correctly, with some candidates simply writing down an expression such as mga or $2mga$ without considering the particular case here. It is probably simpler to note that the centres of gravity of the two rods AB and AC fall vertical distances $\frac{a}{2}$ and $\frac{\sqrt{3}a}{2}$ respectively, though it is also possible to find instead the corresponding distance for the centre of gravity of the system as a whole. A more fundamental fault was to attempt to find the angular acceleration rather than the speed, often wrongly taking it to be constant throughout the motion.

Answer: $\sqrt{\frac{3(1+\sqrt{3})}{14a}}$.

Question 3

The great majority of candidates began by taking moments about A for the system, as suggested in the question, but there were frequent errors in the terms, most commonly in taking the mid-point of the stepladder AB to be a distance $2a$ rather than $\frac{5a}{2}$ from A. The given frictional force at D is then most easily found by taking moments about C for CD, but instead a variety of usually unsuccessful approaches were often seen. The final part follows from noting that the friction at D must be no greater than the product of the coefficient 0.7 and the contact force at D, which leads to an inequality for cot α and hence α. Some candidates reversed this inequality, while others considered equality instead and then concluded that α must be greater than (or less than) the resulting limiting value without stating any justification.

Answers: $\frac{11W}{8}$, $\alpha > 42.3°$.

Question 4

The first of the two given equations follows immediately from the fact that the sum of 0.4cot α and 0.4cot β is equal to the length of AB, namely 1 m. Relating the components of the snooker ball's speeds parallel and normal to the table's side before and after the collision leads to cot β = 4cot α or similar, and substitution for cot β in the first given equation produces the required value of tan α. Most candidates completed at least part of this process, though a quite a few discussed horizontal and vertical motion as if the table were in a vertical plane. Most candidates made reasonable attempts at parts **(ii)** and **(iii)**, using a variety of valid methods.

Answer: **(iii)** 1 m s^{-1}.

Question 5

This seemed to be the least popular of the Mechanics questions, apart from the alternative of the final question, and was not in general well done. The modulus of elasticity may be found by equating the tension to the component of the particle's weight acting along the wire at the equilibrium position and applying Hooke's Law, but it was quite common to see for example mg instead of mgcos 60°. The given SHM equation with $\omega^2 = \frac{2g}{l}$ then follows from applying the equation of motion along the wire at the specified general point. Noting that the speed of the particle when $x = -\frac{1}{4}l$ is given to be $\sqrt{(gl)}$, the standard SHM formula $v^2 = \omega^2(a^2 - x^2)$ yields $a = \frac{3}{4}l$ and hence the maximum speed follows from ωa. The fact that this maximum speed is given in the question may explain why so many candidates were apparently able to write down the amplitude a rather than deriving it. Few candidates were able to find the required time correctly, which follows most easily from $x = a \sin \omega t$ with x here $\frac{1}{4}l$.

Answers: $2mg$, $\sqrt{\frac{l}{2g}}\sin^{-1}\left(\frac{1}{3}\right)$.

Question 6

Most candidates appreciated that the sample mean and estimated population variance must be found and substituted into the usual expression for the confidence interval together with the appropriate critical value for the t-distribution, here 2.776. As usual in such questions, a few candidates were confused over whether to use a biased or unbiased estimate of the population variance, according to whether they had n or $n - 1$ in their version of the confidence interval formula.

Answer: 0.153 ± 0.092 .

Question 7

The appropriateness of the linear model was usually deduced by calculating the product moment correlation coefficient r, here 0.998, though some candidates instead argued convincingly from a scatter diagram or the differences between the successive measurements of v. The question then required the selection of the more suitable regression line, which is here v in terms of θ since the controlled temperature is clearly the independent variable. The other choice of line was often seen, however, perhaps wrongly influenced by the subsequent requirement to estimate θ when $v = 4$. The Examiners felt that the most appropriate comment on the reliability of this estimate is that this point is well outside the range of data and hence unreliable. The frequently stated observation that r is close to 1 is not so relevant to the reliability of this particular point.

Answers: $v = 3.31 + 0.0065\theta$; 106°.

Question 8

Although it was generally realised that the probability of the sample size being less than 12 follows from a geometric distribution, a high proportion of candidates summed one too many terms, finding $1 - q^{12}$ with $q = \frac{7}{8}$ rather than the correct $1 - q^{11}$. The remainder of the question was not often completed correctly. Some candidates found the two separate probabilities for the two alternative sample constituents described in the question, but did not appreciate that these should be summed to give the required overall probability. The expected value may be found by writing down and evaluating the usual summations over r for the two cases, but noting that the term for $r = 1$ is not present in either case so that the values of these terms, here $\frac{7}{8}$ and $\frac{1}{8}$ respectively, must be subtracted from the values of the usual summation over $r = 1, 2,\ldots$ which are here $\left(\frac{7}{8}\right)^{-1}$ and $\left(\frac{1}{8}\right)^{-1}$ respectively. As mentioned in the general comments earlier, some candidates simply wrote down a numerical expression such as $\left(\frac{7}{8}\right)^{-1} + \left(\frac{1}{8}\right)^{-1} - 1$ which, while it yields the correct result, does not really satisfy the question's requirement to show that the given result is correct.

Answers: 0.770; $\frac{7}{8}\left(\frac{1}{8}\right)^{r-1} + \frac{1}{8}\left(\frac{7}{8}\right)^{r-1}$.

Question 9

After stating the null and alternative hypotheses, most candidates rightly attempted to estimate the common variance s^2. A frequent fault was to use 7 and 12 in the numerator rather than 6 and 11. Comparison of the calculated value 2.54 of t with the critical value 2.567 leads to the conclusion that the mean body fat index of Sekitori wrestlers does not exceed that of Makushita wrestlers. Finding the required greatest value of a proved more challenging, requiring that $\dfrac{28.6 - 22.2 - a}{s\sqrt{\frac{1}{7} + \frac{1}{12}}}$ exceeds the critical value 1.740.

Answer: 2.02 .

Question 10

A complete derivation of the distribution function G(*y*) of *Y* involves first integrating f(*x*) between the limits 3 and *x* to find the distribution function F(*x*) = $1 - e^{-(2x - 6)}$, and then demonstrating that G(*y*) is here F(ln *y*). Some candidates used the less convincing approach of simply substituting ln *y* in place of *x* in their expression for F(*x*) without explicit justification. The majority who derived G(*y*) did not appreciate that their expression was capable of simplification, and similarly for the probability density function g(*y*) obtained by differentiation. E(√*Y*) can then be found by integrating g(*y*)√*y* between e^3 and ∞ or equivalently using f(*x*) $e^{\frac{x}{2}}$.

Answers: $1-\frac{e^6}{y^2}$; $\frac{2e^6}{y^3}$; $\frac{4}{3}e^{\frac{3}{2}}$.

Question 11 EITHER

Although this option was attempted almost as often as the Statistics one, many candidates found it challenging, particularly the last two parts. The first part can be approached either by finding the equation of rotational motion for the system of disc and particle and then integrating to obtain the angular velocity, or by using conservation of energy for the latter and then differentiating to find the angular acceleration. In either case the moment of inertia of the system is required, comprising a contribution of ma^2 from both the disc and particle. The given equation involving μ may be proved by first writing down the equations of tangential and radial motion for the particle alone, yielding the frictional and reactive forces *F* and *R* respectively. A good proportion of candidates found the latter correctly, but omitted the term involving the angular acceleration in the former case. Substituting for the angular velocity and acceleration from the first part of the question yields *F* and *R* in terms of θ, and these may in turn be substituted into $F = \mu R$. Showing that the particle cannot lose contact with the disc before it starts to slip was rarely achieved. It is essential to note that contact is lost when $R = 0$. Possible arguments can be based, for example, on the expressions for *F* and *R* in terms of θ, from which it is obvious that as θ increases *F* will reach μR before $R = 0$, or alternatively one can transform the equation given in the question into one involving $t = \tan \frac{1}{2}\theta$ and demonstrate that it will be satisfied by some $t < \tan 30°$.

Answers: $\frac{1}{2}g \sin \theta$, $g(1 - \cos \theta)$.

Question 11 OR

The first of the given expressions for χ^2 is found by substituting into the usual expression for χ^2 and then expanding the numerators of each of the four terms in the summation and simplifying. Substitution of the given values then leads to the second result, and most candidates had little difficulty with this. Writing down some intermediate working, however, such as explaining why $p_3 = 0{\cdot}5 - p$ is advisable in order to demonstrate understanding. The required value p_0 is found by differentiating χ^2 with respect to *p*, setting the resulting expression equal to zero and solving. Some candidates instead wrongly tried to solve $\chi^2 = 0$. Substitution of p_0 yields a value for χ^2 of 7.896, and comparison with the critical value 7.815 leads to the conclusion that the population is not in the claimed proportions. Since any other value of *p* cannot yield a smaller value of χ^2 than 7.896, the conclusion must be unaltered.

Answer: 0.198 .