## Section 2 of Unit 03 (Pure Mathematics 3)

## Calculus

**Recommended Prior Knowledge**
Students must have covered the work on Calculus and Coordinate Geometry from P1 and the work on partial fractions and the exponential and logarithmic functions in Section1 of the Unit P3 (Topic 6) .

**Context**
This Section extends the work on calculus studied in P1. Because of the importance of a good understanding of calculus, this work should be covered relatively early in the course, though it is also essential that the work on the exponential and logarithmic functions has already been thoroughly studied. It is suggested that the work on integration using trigonometric identities be left until later in the course when the trigonometry in Section 3 has been completed.

**Outline**
The work on Calculus covered in P1 is continued and extends to the differentials of the three basic trigonometric functions and the exponential and logarithmic function. The differential of the product and quotient of two functions is studied, along with the first derivative of functions that are expressed either parametrically or implicitly. Integration as the reverse of differentiation is applied where appropriate to all the basic differentials already covered. Integration by parts, by substitution and by partial fractions is also covered in detail. This Section finishes by looking at the use of the trapezium rule for numerical integration.

| Topic | Learning Outcomes | Suggested Teaching Activities | Resources | On-Line Resources |
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| 4 | **Differentiation**<br>• Use the derivatives of $e^x$, ln$x$, sin$x$, cos$x$, tan$x$, together with constant multiples, sums, differences, and composites. | The basic results for the derivatives of the three trigonometric functions can either be stated or verified by calculating gradients of tangents on accurate graphs (it will be necessary to define the secant function at this point if Topic 3 has not been completed). It is necessary to emphasise to students that the results are correct only when the angle is in radians. OHP graphs showing these basic results are particularly useful. The definitions of the exponential and logarithmic functions were met in Section 1; their derivatives should now be introduced and used. There is a need for a lot of practice using the derivatives of multiples and composites of all these trigonometric, exponential and logarithmic functions. | OHP graphs showing the derivatives of the basic trigonometric functions. | www.bbc.co.uk/education/asguru/maths<br>→pure<br>→Differentiation<br>→Differentiation of $e^x$ and ln$x$. |

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| | • Differentiate products and quotients.<br><br>• Find and use the first derivative of a function which is defined parametrically. | Extend the work to stating the formulae for the derivatives of products and quotients of two basic functions. It is worth suggesting that brighter students look at the proofs from first principles found in any A Level text-book. All the work on differentiation should be revised using the additional formulae now available e.g. finding stationary points, equations of tangents and normals for curves of the type $y=x^2e^{2x}$, y= $x\ln x$, $y=\frac{ax+b}{cx+d}$, $y=x\sin x$ etc.<br><br>The idea of expressing a function by means of two parametric equations will be a new one for most students. Sketching accurately ( a good exercise for homework) curves such as " $x = 2t+3$, $y = t - 4$ " or "$x = t^2$, $y = t - 2$" or " x=2sint , y=2cost" is a worthwhile exercise, and it is recommended that such accurate graphs are available on OHPs. If time allows, questions using the principles from the coordinate geometry of P1 can be repeated with curves expressed in parametric form. Students should also be able to find the Cartesian form of functions expressed simply in parametric coordinates. Discuss fully with students, by reference to different examples, the derivative of a function that is defined parametrically. Use derivatives to find stationary values and where appropriate the greatest value of $x$, the equations of tangents and normals and other problems using the coordinate geometry principles in P1. Students should be able to obtain the general form of the equation of the tangent and normal in the form $y=m(t)x+c(t)$ where $t$ is the parameter. Students are not expected to be able to obtain a second derivative from parametric equations. | OHP showing all the derivatives from P1 and P2.<br><br>OHP showing an accurate graph obtained by plotting with parametric coordinates. | |

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| | • Find and use the first derivative of a function which is defined implicitly. | Discuss with students the idea of "explicit" and "implicit" functions. Show students that the derivative, with respect to *x*, of say $y^2$ can be obtained by use of the chain rule.<br>Discuss the ideas behind differentiating, say, $x^2 + y^2 = 100$.<br>Look specifically at the use of the product rule for the derivative of, say, "*xy*",<br><br>Students will need plenty of practice working with the derivatives of all the functions encountered in this section. In particular, they should be able to use derivatives to find stationary values, the equations of tangents and normals and other problems using the coordinate geometry principles in P1. | | |
| 5 | **Integration**<br>• Extend the idea of "reverse differentiation" to include the integration of $e^{ax+b}$, $\frac{1}{ax+b}$, $\sin(ax+b)$, $\cos(ax+b)$, $\sec^2(ax+b)$ | Students should obtain for themselves, the basic integrals of $e^x$, $\frac{1}{x}$, sin*x*, cos*x* and $\sec^2 x$. Discussion should then follow on finding the integrals of these functions when "*x*" is replaced by "*ax+b*". Students should realise that any rules formulated do not apply to functions other than those in which *x* is replaced by a linear function.<br><br>Discuss with students problems met in Unit 2 requiring integration i.e. finding the equation of a curve from a gradient and finding area s and volumes of revolution, but now using functions new to this Unit.<br><br>Particular problems to investigate include finding the area under a curve of the type $y = a\sin x + b\cos x$, or $y = \frac{12}{2x-3}$, or $y = e^x + e^{-x}$ etc. or finding the volume under curves of the type $y = \frac{4}{ax+b}$ or $y = \frac{2}{\cos(ax+b)}$. | OHP showing a full list of all the integrals in P1 and P2. | www.bbc.co.uk/education/asguru/maths<br>→pure<br>→integration of $e^x$ and ln*x*. |

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| | ▪ Use trigonometrical relationships (such as double-angle formulae) to facilitate the integration of functions such as $\cos^2 x$.<br><br>• Integrate rational functions by means of decomposition into partial fractions (restricted to the types of partial fractions specified in Topic 1 of P3) | Students should be made aware that certain functions can be integrated by using an appropriate trigonometric formula. This section must however be left until after the work on trigonometry in topic 3 has been completed. It is appropriate to return to this part of topic 5 at a later stage (revising at the same time).<br><br>Students should be shown that<br>• the integral of both $\sin^2 x$ and $\cos^2 x$ can be achieved by the appropriate use of the formula for $\cos 2x$<br>• the integral of $\sin x \cos x$ can be achieved by the use of the formula for $\sin 2x$<br>• the integral of $\tan^2 x$ ( and hence the volume of rotation of $y = \tan x$) can be obtained from the formula linking $\sec^2 x$ and $\tan^2 x$.<br><br>Students should be encouraged to suggest a method for integrating such functions as $\frac{12}{(x+1)(x+2)}$. Hopefully, many will recognise the need to express such functions in partial fractions. This is a suitable time to revise the section on partial fractions from the Algebra topic. Students will realise that the following partial fractions will be obtained:<br><br>• $\frac{A}{ax+b}$,<br>• $\frac{A}{(ax+b)^2}$,<br>• $\frac{Ax}{x^2+c^2}$. | OHP showing the types of partial fractions in this syllabus. Also worth having an OHP showing examples of the use of partial fractions in integration. | |

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| | | Students should also realise that each of these functions can be integrated by one of the methods in this topic. Teachers should note that ideally, it is necessary to complete the sections on recognition of an integrand as $\frac{kf'(x)}{f(x)}$ and the section on integration by substitution before students are able to integrate all of the partial fractions listed above. Lots of practice is to be recommended, including the use of partial fractions to integrate the function $\frac{1}{1-x^2}$. | | |
| | ▪ Recognise an integrand of the form $\frac{kf'(x)}{f(x)}$, and integrate, for example, $\frac{x}{x^2+1}$ or tan*x*. | Students should be encouraged to look at the integration of a simple function such as $\frac{2x}{x^2+1}$. Hopefully a few will recognise that since the numerator is the differential of the denominator, then the answer is ln $(x^2+1)+c$. (It is also worth revising the differentiation of the function ln f(*x*)). This work should then be extended to such integrands as $\frac{x}{x^2+1}$ and $\frac{5x}{x^2+1}$ etc. Other similar functions such as tan*x* $\frac{x^2}{x^3+2}$, $\frac{1}{x\ln x}$ and $\frac{1}{2+e^{-2x}}$ $(\equiv \frac{e^{2x}}{2e^{2x}+1})$ should be covered in a similar way. | OHP showing the differentiation of a function such as $\ln(x^2+1)$ and its use in integrating $\frac{x}{x^2+1}$. | |

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| | • Recognise when an integrand can usefully be regarded as a product and use integration by parts to integrate, for example, $x\sin 2x$, $x^2e^x$ or $x\ln x$. | Students should also appreciate why this method cannot be used with such functions as $\frac{1}{x^2+1}$ and $\frac{x}{x^3+1}$.<br><br>Following revision of the differentiation of a product, the result "$\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x)=\sin x+x\cos x$" should be obtained. Students should then be encouraged to realise that integration of both sides of this equation leads to the result "$x\sin x=\int\sin x\mathrm{d}x+\int x\cos x\mathrm{d}x$" and that this result provides a method for integrating the product $x\cos x$. The general formula for integration by parts can then be deduced. Students should be given the opportunity for using this method to integrate the functions $x\sin x$, $x\cos x$ and $xe^x$. The integration of more complex functions such as $x\sin 2x$ or $xe^{-2x}$ should also be considered. Students should be shown that:<br>• application of the process twice enables them to integrate such functions as $x^2\sin x$<br>• the same method can be applied to $x\ln x$ but that in this case the "$x$" should be used as the "*v*' " in the general formula<br>• that this approach enables all functions of the form $x^n\ln x$, where *n* is an integer, to be integrated, including the special case $\ln x$. | OHP showing the differentiation of a product such as $x\sin x$ and its use in the integration of $x\cos x$.<br><br>OHP showing the general formula and how the formula applies differently for $xe^x$ and for $x\ln x$. | |

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| | • Use a given substitution to simplify and evaluate either a definite or an indefinite integral.<br><br>• Use the trapezium rule to estimate the value of a definite integral, and use sketch graphs in simple cases to determine whether the trapezium rule gives an over-estimate or an under-estimate. | Students should be able to integrate such functions as $\frac{8}{(x+1)^3}$ or $\sqrt{(3x+5)}$ or $\frac{x}{x^2+1}$ by the methods encountered so far in this section. They should now be encouraged to use the method of substitution, using a substitution (say, u), which will be given, to integrate each of the above expressions. The need to obtain an expression for $\frac{\mathrm{d}u}{\mathrm{dx}}$ so that the "d*x*" of the original integral can be expressed in terms of "d*u*" must be stressed. Students should be able to evaluate both definite and indefinite integrals. They should be shown that a definite integral can be evaluated by using either the original variable with the original limits of integration or by using the new variable providing that the limits are changed correspondingly. Considerable practice is needed to enable the students to become familiar with change of variable and it is worth being particular strict on ensuring correct notation. Students will soon become aware of the need to include the "d*x*" with the integration.<br><br>Some students will have met the idea of using a numerical method for finding the approximate area under a curve (maybe in physics with the distance travelled as the area under a v-t graph or in other maths syllabuses for finding the area below a curve). | OHP showing the integration of a definite integral. The slide should show that the same answer can be achieved using either the original variable (*x*) or the new variable (*u*).<br><br>OHP slide showing the accurate graph of *y*= $\sqrt{2x^2+1}$ . This should also show the approximate answers for the area when 2, 3, 4 or more strips are considered. | |

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| | | Discussion should commence with a problem such as $\int_0^2 \sqrt{2x^2+1}\,\mathrm{d}x$. Students should recognise that this cannot be achieved by normal methods of integration. They should be encouraged to sketch the graph of y= $\sqrt{2x^2+1}$ and to suggest methods for finding an approximate value of the integral (have ready an OHP slide showing an accurate graph of y= $\sqrt{2x^2+1}$ ).They should appreciate that by taking more and more trapezia, more accurate results can be obtained. Although formal proof of the formula for using the trapezium rule is not required, most groups will appreciate and be able to understand it.<br><br>Students will need practice at using the rule and should be aware, from appropriate sketch graphs, as to whether the approximate answer is an under-estimate or an over-estimate of the true result. This is an appropriate place to use the OHP slide for *y*= $\sqrt{2x^2+1}$ to show whether an answer is an over-estimate or an under-estimate. Have other examples ready for the students along with corresponding OHP slides. These should include looking at $\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$ and showing that the trapezium rule underestimates the exact value, and that a sequence of estimates obtained from increasing the number of strips of equal width (say *n*=1, 2, 4, 8…) converges to ¼π. | Other OHP slides showing cases of “over-estimates” or “under-estimates”. | |

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| 8 | **Differential Equations**<br>• Formulate a simple statement involving a rate of change as a differential equation including the introduction if necessary of a constant of proportionality. | A good starting point to this topic is to discuss with students a statement such as "The number of cases of a disease is increasing at a rate proportional to the number of cases recorded". Although this statement is much simplified (the actual equation is more complex), it is an example that most students find particularly interesting and is an excellent example from which to begin the teaching of differential equations. This will enable teachers to introduce the idea of a differential equation being basically an equation (with an "=" sign) and involving a rate of change. It is worth reminding students of the exact meaning of $\frac{\mathrm{d}y}{\mathrm{dx}}$ or $\frac{\mathrm{d}N}{\mathrm{dt}}$. They should realise that the idea of "proportional to" involves a constant *k*. Most students should then be able to translate the above statement as "$\frac{\mathrm{d}N}{\mathrm{dt}} = kN$". Students should be given the opportunity to convert other statements into differential equations, including the process of radioactive decay leading to the equation "$\frac{\mathrm{dm}}{\mathrm{dt}} = -km$". | The problem outlined here (cases of disease with original conditions) should be available on an OHP along with the complete solution. This should show the various steps of the calculation :- formulation, separation, integration, evaluation of *k* and *C,N* as function of *t*. | |
| | • Find by integration a general form of solution for a first order differential equation in which the variables are separable. | Following the work on integration by substitution, students should realise that the equation $\frac{\mathrm{d}N}{\mathrm{dt}} = kN$ can be rewritten as $\int \frac{\mathrm{d}N}{\mathrm{N}} = \int k\mathrm{d}t$ (the process being known as separating the variables). They should be able to integrate this equation to obtain $\ln N = kt + C$ (known as the general form of solution). It is essential that the students should now realise the need to include the constant of integration. Lots of practice can now be given on other basic differential equations. | | |

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| | • Use an initial condition to find a particular solution.<br><br>• Interpret the solution of a differential equation in the context of a problem being modelled by the equation | Returning to the original statement of "The number of cases of a disease is increasing at a rate proportional to the number of cases recorded", original conditions can be introduced. For example, when t = 0 (days), the number of recorded cases is 20 and when t = 7 (days), the number of recorded cases is 50.<br><br>Students should be able to use these conditions in the solution obtained above, i.e. "$\ln N = kt + C$", to obtain values of *k* and *C* and hence to find a particular solution for this problem.<br><br>Candidates should also be encouraged to express this solution in its simplest form with *N* as the subject, as well as with ln *N* as the subject. They should be encouraged to plot the graph of *N* against *t* for this problem and to interpret results in a practical way.<br><br>Similar problems with other situations, (Newton's Law of cooling, population growth, radioactive decay etc) should be discussed with the students. | | |