Section 1 of Unit 03 (Pure Mathematics 3)

Algebra

Recommended Prior Knowledge
Students should have studied the algebraic techniques in Pure Mathematics 1.

Context
This Section should be studied early in the course since the work on exponential and logarithmic functions, and on partial fractions, is needed for Section 2.

Outline
The Section solves equations and inequalities involving the modulus function and examines the use of the factor and remainder theorems in solving cubic and quartic equations. Work on the binomial expansion $(1+x)^n$, is extended to consider all values of n. The Section deals with the decomposition of rational functions into partial fractions (Topic 1). The exponential and logarithmic functions are introduced. Logarithms are used to solve equations of the form $a^x = b$ and to reduce equations to linear form (Topic 2). The Section concludes by dealing with the numerical solution of equations by use of an iterative formula (Topic 6).

Topic	Learning Outcomes	Suggested Teaching Activities	Resources	On-Line Resources
1	**Algebra** • Understand the meaning of $\lvert x\rvert$ and use relations such as $\lvert a\rvert = \lvert b\rvert \Leftrightarrow a^2 = b^2$ and $\lvert x - a\rvert \le b \Leftrightarrow a - b \le x \le a + b$ in the course of solving equations and inequalities.	General discussion on the modulus function. Students must appreciate that $\lvert 4\rvert = \lvert -4\rvert = 4$. They should be encouraged to draw graphs of functions such as $y = \lvert 2x - 3\rvert$, recognising the "V" shape of the graph and the point (1.5,0) on the *x*-axis. They should be encouraged to sketch graphs of the modulus of other functions such as simple trigonometric graphs and quadratics with stationary points below the *x*-axis. Students should be appreciative of the fact that $\lvert a\rvert = \lvert b\rvert \Leftrightarrow a^2 = b^2$ and be able to use these as appropriate for solving equations such as $\lvert 2x - 3\rvert = 7$. The graphical solution of this equation should also be examined, as should the alternative method of equating (2*x*-3) to ± 7. Students should realise that the solution of the inequality $\lvert 2x - 3\rvert < 7$ can be obtained either from $(2x - 3)^2 < 49$ or from $-7 < 2x - 3 < 7$.	OHP with graphs of such functions as $y = \lvert \sin x\rvert$, $y = \lvert x^2 - 4\rvert$ and $y = \lvert 3x - 5\rvert$.	www.bbc.co.uk/education/asguru/maths →pure →functions →modulus

	• Divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero). Use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients.	The process of dividing a polynomial (cubic or quartic) by either a linear or quadratic polynomial should be discussed (an OHP is particularly useful). Practice exercises should include cases in which a polynomial contains a coefficient of zero (for example the division of a quartic that has no coefficient in x^3). The alternative method of avoiding the long division by comparing coefficients should also be examined, especially with more able groups. Students should be aware that, in the case of division of f(x) by, say, (x – 3), the remainder is f(3).This answer should be verified by the division process. This leads naturally to the factor theorem – i.e. if the remainder is zero when dividing by (x-3), then (x-3) is a factor of f(x) and that x=3 is a solution of f(x)=0. Students should be able to factorise polynomial expressions and to solve polynomial equations by first using trial and error to find a solution or factor and then completing the process by division.	OHP showing in detail the process of long division of a cubic polynomial with a linear polynomial.	
	• Recall an appropriate form for expressing rational functions in partial fractions, and carry out the decomposition, in cases where the denominator is no more complicated than $(ax+b)(cx+d)(ex+d)$ $(ax+b)(cx+d)^2$ $(ax+b)(x^2+c^2)$ and where the degree of the numerator does not exceed that of the numerator.	As an introduction to partial fractions, it is a worthwhile exercise revising some of the basic algebra from O Level or IGCSE by asking the students to express the following as single fractions:- • $2-\frac{4}{x+2}$ • $\frac{3}{x-2}+\frac{5}{x+3}$ • $\frac{2}{(x+2)}+\frac{3}{x-1}$ • $\frac{2x}{x^2+4}-\frac{3}{x-2}$	OHP showing the process needed at O Level or at IGCSE for the addition of two or more partial fractions to form a single fraction.	

		The question of reversing the process can then be introduced. Students should be encouraged to realise that a rational function such as $\frac{12}{(x+2)(x-1)}$ can be reduced to two partial fractions of the form $\frac{A}{x+2}+\frac{B}{x-1}$. Students should be encouraged to reach the identity $12 \equiv A(x-1)+B(x+2)$ for themselves. They should be shown that such an identity holds for all values of *x* and that substitution of any values of *x*, but particularly *x = 1* and *x* = -2, will enable the values of *A* and *B* to be found. They should appreciate why these two particular values of *x* have been selected. They should also appreciate that equations for *A* and *B* can be obtained by comparing the coefficients of *x* and, in this case, the constant on each side of the identity. Similar processes should enable the students to express such rational functions as $\frac{6x^2}{(x-1)^2(x+2)}$ and $\frac{12}{x(x^2+4)}$ in partial fractions. Students should be made aware of the need to divide first in cases where the degree of the numerator is the same as that of the denominator. In such cases the division can often be done synthetically more easily that formally, especially with such fractions as $\frac{x^2}{x^2+x-2}$. They should be encouraged to commit to memory all the possible decompositions within the syllabus.	OHP showing all the possible decompositions within this syllabus.	

	• Use the expansion of $(1+x)^n$, where *n* is a rational number and $\|x\| \prec 1$ (finding a general term is not included, but adapting the standard series to $(2-\frac{1}{2}x)^{-1}$ is included)	The work in Topic 7 (Series) on Unit P1 should be revised. Students should be comfortable in finding expressions for $(1+x)^n$ for simple integral values of *n*. They should be aware that the coefficient of x^r in this expansion can be obtained from the recurrence relation ${}_nC_{r+1} = \frac{n-r}{r+1}\,{}_nC_r$ They should be made aware that this same expansion holds for values of *n* that are non-integral, providing that $\|x\|<1$. Students should realise that the expansion of $(1+x)^n$ can be similarly used with such series as $\sqrt{1+2x}$ with "*x*" being replaced by "2*x*" and *n* = ½ and for $\frac{6}{(1-2x)^3}$ with "*x*" being replaced by "−2x" and *n* = −3. Lots of practice is needed. Students then need to realise that the expansion of $(2-\frac{1}{2}x)^{-1}$ can also be obtained by rewriting it as $2^{-1}(1-\frac{1}{4}x)^{-1}$ and then using the basic expansion of $(1+x)^n$ with "*x*" replaced by " −¼*x*" and *n* = −1.		

2	**Logarithmic and Exponential Functions** • Understand the relationship between logarithms and indices, and use the laws of logarithms.	General discussion on the definition of a "logarithm". Show students that the statement "10^2=100" can be written as "$\log_{10}100 = 2$". If time allows, show students how multiplication and division was carried out in "pre-calculator" days by using logarithmic tables. Show students, again on the calculator, that "$\log_{10}2$=0.301" is equivalent to "$10^{0.301}$=2". Illustrate, on the calculator, why it is that "2×3 = 6" is equivalent to the addition "$\log_{10}2+\log_{10}3=\log_{10}6$" and give similar results for division and for exponential calculations. Deduce the laws of logarithms and give the students plenty of practice in • evaluating such expressions as $\log_2 8$ etc, • writing expressions such as $3\log_k a+\log_k b-\log_k c$ as a single logarithm and vice versa	OHP with examples of the logarithmic calculations used in the time before the calculator appeared in schools. Include in this an explanation of why the process works, referring back to powers of 10.	www.bbc.co.uk/education/asguru/maths →pure →exponentials and logarithms
	• Understand the definition and properties of e^x and ln x, including their relationship as inverse functions and their graphs.	Discuss the properties of the graphs of $y = e^x$ and $y = \ln x$. Draw accurate graphs and use these to deduce that the gradient of the curve $y = e^x$ is equal to e^x and that the gradient of the curve $y = \ln x$ is equal to x^{-1} (this makes an excellent piece of homework). These results can be held back until the next section (Calculus) is covered, but it is worth pointing out that the gradient of e^x is always positive and always increasing and that the gradient of ln$x$ is always positive and always decreasing. Students should be aware that the functions e^x and ln x are inverse functions (check on the calculator) and that the graphs of $y = e^x$ and $y = \ln x$ are mirror images in the line $y = x$. Students should be aware that $\ln(e^x) = x$ and that $e^{\ln x}=x$. It should be pointed out that the range of $f : x \mapsto e^x$ for $x \in R$ is $x \in R^+$ and that the range of $f: x \mapsto \ln x$ for $x \in R^+$ is $x \in R$.	OHP showing the accurate graphs of $y=e^x$ and $y=\ln x$ and illustrating the inverse property (reflection in the line $y = x$) and showing how the gradient functions of each can be verified.	

	• Use logarithms to solve equations of the form $a^x = b$, and similar inequalities. • Use logarithms to transform a given relationship to linear form, and hence determine unknown constants by considering the gradient and / or intercept.	Students should consider the solution of the equation $a^x = b$ and should be encouraged to obtain a solution by trial and improvement. They should be shown how expressing the equation in logarithmic form reduces it to a linear equation. Students should be able to solve equations such as $2^{x+1} = 3^x$, and be able to extend the work to inequalities. They need to be aware that the solution of such inequalities as $(0.6)^n < 0.2$, results in a solution $n > k$, rather than $n < k$. (they should appreciate that log 0.6 is negative and that division by a negative quantity affects the sign of the terms in the inequality). Such questions might be posed in the form "find the smallest value of *n* for which the *n*th term of the geometric progression with first term 2 and common ratio 0.9 is less than 0.1". Students should be able to convert: • (i) the equation $y = ax^n$ to logarithmic form recognising that the resulting equations is linear, giving a straight line graph when log*y* is plotted against log*x*, • (ii) the equation $y = A(b^x)$ to logarithmic form recognising that the resulting equation is linear, giving a straight line graph when log*y* is plotted against *x*. • (iii) the equation $y = Ae^{nx}$ to the form $\ln y = \ln A + nx$ recognising that this form is linear, giving a straight line when ln*y* is plotted against *x*. Students should have plenty of practice in deducing the values of either *a* and *n* in (i) or *A* and *b* in (ii) or A and n in (iii) by drawing straight line graphs from given experimental values of *x* and *y*. Such exercises should include cases in which the variables have symbols other than *x* and *y* (representing, for example, particular physical quantities).	OHP showing values of $(0.6)^n$ for different *n* to illustrate that $n>k$ rather than $n<k$. OHP graphs for the linear forms of the equations $y = ax^n$ $y = A(b^x)$ and $y = Ae^{nx}$.	

6	**Numerical Solution of Equations** • Locate approximately a root of an equation, by means of graphical considerations and / or searching for a sign change.	Students will have experienced the solution of equations where methods can be used to give exact answers – linear equations, quadratic equations, trigonometric equations, equations of the form $2^x = k$ etc. It is worth recalling all of these techniques. General discussion should follow on the solution of such equations as, for example, (i) $x^3 + 5x = 100$, (ii) $x^2 = \cos x$ and (iii) $e^x = \sqrt{x+3}$. Students should have used the method of trial and improvement for solving equations of type (i) and should realise that a sketch of $f(x) = x^3 + 5x - 100$ (or sketches of $y = x^3$ and $y = 100 - 5x$), or a search for a sign change, will lead them to a solution in the range $x = 4$ to $x = 5$. In case (ii), sketches of $y = x^2$ and $y = \cos x$ should lead to the conclusion that there is only one positive root and that the root lies in the range $0 < x < \frac{\pi}{2}$. Similarly sketches of $y = e^x$ and $y = \sqrt{x+3}$ (or $y = e^{2x}$ and $y = x + 3$) for case (iii) should lead to the conclusion that there is only one positive root and that the root lies in the interval $0 < x < 1$. Students will need a lot of practice with similar equations, including cases in which some manipulation may be necessary to obtain two suitable functions e.g. find the smallest root of $e^x \sin x = 1$.	OHP showing each of the 3 graphs used in examples (i), (ii) and (iii).	www.bbc.co.uk/education/asguru/maths →pure →numerical methods
	• Understand the idea of, and use the notation for, a sequence of approximations which converge to a root of an equation.	The idea of an iterative formula will probably have been used in topic 7 of Unit P1 with arithmetic and geometric progressions ($x_n = a + (n-1)d$ and $x_n = ar^{n-1}$). Give students practice in using equations of the form $x_n = f(n)$ and of the form $x_{n+1} = F(x_n)$.		

	• Understand how a given simple iterative formula of the form $x_{n+1} = \mathrm{F}(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine the root to a prescribed degree of accuracy. (Knowledge of the condition for convergence is not included, but candidates should understand that an iteration may fail to converge).	Discuss the meaning of an equation of the type $x_{n+1} = \sqrt[3]{x_n + 10}$. Show students that using an initial value for x_1 of either 2 or 3 will lead to a series of numbers that converge eventually to 2.31. This is the ideal place to discuss simple ideas of convergence and divergence. Students should be aware of the need to write down the result of each iteration, usually to one more place of accuracy than requested for the final answer. Students should be aware that the value of the previous iteration need not be keyed in to the calculator explicitly at each stage; the "ans" key can be used instead. Students should be given plenty of practice on this procedure. Discuss with the students the technique of setting $x_{n+1} = x_n = L$ (or other convenient letter) in the case of convergence, and using this to deduce the equation for which the result of the iteration is a root. Expressing an equation of the form $x^3=x+10$ as "$x=$" can be done in many ways, each leading to an iterative formula of the type $x_{n+1} = \mathrm{F}(x_n)$. It is a worthwhile exercise having several of these different iterative formulae available on an OHP and showing students that some of them converge, but that in some cases, the series diverges. The students do not need to know of conditions for convergence. It is also worth demonstrating that a particular iterative formula may give different limiting values for different values of x_0, e.g. the formula $x_{n+1} = \frac{x_n^2 - 1}{2x_n - 4}$ with x_0=0.25 and with x_0=3.75.	OHP showing different iterative formulae for the solution of $x^3=x+10$.	