UNIT 4 Algebra (P2)

Recommended Prior Knowledge. Students should have studied the algebraic techniques in Unit 1 (Algebra) from Pure Mathematics 1.

Context. This Unit should be studied before Unit 5 in P2 since the work on exponential and logarithmic functions is needed for Unit 5. The order of Topics within this Unit should be either " Topic 1, 2 and 6" or "Topic 1, 6, 2".

Outline. The unit solves equations and inequalities involving the modulus function and examines the use of the factor and remainder theorems in solving cubic and quartic equations (Topic 1). It introduces the exponential and logarithmic functions and looks at the use of logarithms in solving equations of the form $a^x = b$ and in reducing equations to linear form (Topic 2). It concludes by looking at the numerical solution of equations by use of an iterative formula (Topic 6). The latter part of Topic 1 makes an ideal introduction to Topic 6.

Topic	Learning Outcomes	Suggested Teaching Activities	Resources	On-Line Resources
1	**Algebra** Understand the meaning of $\|x\|$ and use relations such as $\|a\| = \|b\| \Leftrightarrow a^2 = b^2$ and $\|x - a\| \le b \Leftrightarrow a - b \le x \le a + b$ in the course of solving equations and inequalities.	General discussion on the modulus function. Students must appreciate that $\|4\| = \|-4\| = 4$. They should be encouraged to draw graphs of functions such as $y = \|2x - 3\|$, recognising the "V" shape of the graph and the point (1.5,0) on the *x*-axis. They should be encouraged to sketch graphs of the modulus of other functions such as simple trigonometric graphs and quadratics with stationary points below the *x*-axis. Students should be appreciative of the fact that $\|a\| = \|b\| \Leftrightarrow a^2 = b^2$ and be able to use these as appropriate for solving equations such as $\|2x - 3\| = 7$. The graphical solution of this equation should also be examined, as should the alternative method of equating (2*x*-3) to ± 7. Students should realise that the solution of the inequality $\|2x - 3\| < 7$ can be obtained either from $(2x - 3)^2 < 49$ or from $-7 < 2x - 3 < 7$.	OHP with graphs of such functions as $y = \|\sin x\|$ or $y = \|x^2 - 4\|$ or $y = \|3x - 5\|$ etc.	www.bbc.co.uk/education/asguru/maths →pure →functions →modulus

	Divide a polynomial, of degree not exceeding 4, by a linear or quadratic polynomial, and identify the quotient and remainder (which may be zero). Use the factor theorem and the remainder theorem, e.g. to find factors, solve polynomial equations or evaluate unknown coefficients.	The process of dividing a polynomial (cubic or quartic) by either a linear or quadratic polynomial should be discussed (an OHP is particularly useful) and plenty of practice given. The alternative method of avoiding the long division by comparing coefficients should also be examined, especially with more able groups. Students should be aware that, in the case of division of f(x), by, say, (x – 3), the remainder is f(3).This answer should be verified by the division process. This leads naturally to the factor theorem – i.e. if the remainder is zero when dividing by (x-3), then (x-3) is a factor of f(x) and that x=3 is a solution of f(x)=0. Students should be able to factorise polynomial expressions and to solve polynomial equations by finding, by trial and error, a first solution or factor followed by the resulting quadratic (or cubic) obtained by division.	OHP showing in detail the process of long division of a cubic polynomial with a linear polynomial.	
2	**Logarithmic and Exponential Functions** Understand the relationship between logarithms and indices, and use the laws of logarithms.	General discussion on the definition of a “logarithm”. Show students that the statement “$10^2=100$” can be written as “$\log_{10}100 = 2$”. If time allows, show students how multiplication and division was carried out in “pre-calculator” days by using logarithmic tables. Show students, again on the calculator, that “$\log_{10}2=0.301$” is equivalent to “$10^{0.301}=2$”. Illustrate, on the calculator, why it is that “$2\times3 = 6$” is equivalent to the addition “$\log_{10}2+\log_{10}3=\log_{10}6$” and give similar results for division and for exponential calculations. Deduce the laws of logarithms and give the students plenty of practice in • evaluating such expressions as $\log_2 8$ etc, • writing expressions such as $3\log_k a+\log_k b-\log_k c$ as a single logarithm and vice versa	OHP with examples of the logarithmic calculations used in the time before the calculator appeared in schools. Include in this an explanation of why the process works, referring back to powers of 10.	www.bbc.co.uk/education/asguru/maths →pure →exponentials and logarithms

	Understand the definition and properties of e^x and ln *x*, including their relationship as inverse functions and their graphs.	Discuss the properties of the graphs of $y = e^x$ and $y = \ln x$. Draw accurate graphs and use these to deduce that the gradient of the curve $y = e^x$ is equal to e^x and that the gradient of the curve $y = \ln x$ is equal to x^{-1} (this makes an excellent piece of homework). These results can be held back until the next Unit (Calculus) is covered, but it is worth pointing out that the gradient of e^x is always positive and always increasing and that the gradient of ln*x* is always positive and always decreasing. Students should be aware that the functions e^x and ln *x* are inverse functions (check on the calculator) and that the graphs of $y = e^x$ and $y = \ln x$ are mirror images in the line $y = x$. Students should be aware that $\ln(e^x) = x$ and that $e^{\ln x} = x$. It should be pointed out that the range of $f : x \mapsto e^x$ for $x \in R$ is $x \in R^+$ and that the range of $f: x \mapsto \ln x$ for $x \in R^+$ is $x \in R$.	OHP showing the accurate graphs of $y=e^x$ and $y=\ln x$ and illustrating how the gradient functions of each can be verified.	
	Use logarithms to solve equations of the form $a^x = b$, and similar inequalities.	Students should look at the solution of the equation $a^x = b$. They should be encouraged to obtain a solution by trial and improvement. They should be shown how expressing the equation in logarithmic form reduces it to a linear equation. Students should be able to solve similar equations such as $2^{x+1} = 3^x$, and be able to extend the work to inequalities. They need to be aware that the solution of such inequalities as $(0.6)^n < 0.2$, results in a solution $n > k$, rather than $n < k$. (they should appreciate that log 0.6 is negative and that division by a negative quantity affects the sign of the inequality). Such questions might be posed in the form "find the smallest value of *n* for which the *n*th term of the geometric progression with first term 2 and common ratio 0.9 is less than 0.1".	OHP showing values of $(0.6)^n$ for different *n* to illustrate that $n>k$ rather than $n<k$.	
	Use logarithms to transform a given relationship to linear form, and hence	Students should be able to convert (i) the equation $y = ax^n$ to logarithmic form recognising that the resulting	OHP graphs for the linear forms of the	

	determine unknown constants by considering the gradient and / or intercept.	equations is linear, giving a straight line graph when log*y* is plotted against log*x*, and (ii) the equation $y = A(b^x)$ to logarithmic form recognising that the resulting equation is linear, giving a straight line graph when log*y* is plotted against *x*. Students should have plenty of practice in deducing the values of either *a* and *n* or *A* and *b* by drawing straight line graphs from given experimental values of *x* and *y*. Such exercises should include cases in which the variables have symbols other than *x* and *y* (representing, for example, particular physical quantities).	equations $y = ax^n$ and $y = A(b^x)$.	
6	**Numerical Solution of Equations** Locate approximately a root of an equation, by means of graphical considerations and / or searching for a sign change.	Students will have experienced the solution of equations where methods can be used to give exact answers – linear equations, quadratic equations, trigonometric equations, equations of the form $2^x = k$ etc. It is worth recalling all of these techniques. General discussion should follow on the solution of such equations as, for example, (i) $x^3 + 5x = 100$, (ii) $x^2 = \cos x$ and (iii) $e^x = \sqrt{x+3}$. Students should have used the method of trial and improvement for solving equations of type (i) and should realise that a sketch of $f(x) = x^3 + 5x - 100$, or a search for a sign change, will lead them to a solution in the range *x* = 4 to *x* = 5. In case (ii), sketches of $y = x^2$ and $y = \cos x$ should lead to the conclusion that there is only one positive root and that the root lies in the range $0 < x < \frac{\pi}{2}$. Similarly sketches of $y = e^x$ and $y = \sqrt{x+3}$ for case (iii) should lead to the conclusion that there is only one positive root and that the root lies in the interval $0 < x < 1$. Students will need a lot of practice with similar equations, including cases in which some manipulation may be necessary to obtain two suitable functions e.g. find the smallest root of $e^x \sin x = 1$.	OHP showing each of the 3 graphs used in examples (i), (ii) and (iii).	www.bbc.co.uk/education/asguru/maths →pure →numerical methods

	Understand the idea of, and use the notation for, a sequence of approximations which converge to a root of an equation. Understand how a given simple iterative formula of the form $x_{n+1} = \mathrm{F}(x_n)$ relates to the equation being solved, and use a given iteration, or an iteration based on a given rearrangement of an equation, to determine the root to a prescribed degree of accuracy. (Knowledge of the condition for convergence is not included, but candidates should understand that an iteration may fail to converge).	The idea of an iterative formula will probably have been used in Unit 1 in P1 with arithmetic and geometric progressions ($x_n = a + (n-1)d$ and $x_n = ar^{n-1}$). Give students practice in using equations of the form $x_n = \mathrm{f}(n)$ and of the form $x_{n+1} = \mathrm{F}(x_n)$. Discuss the meaning of an equation of the type $x_{n+1} = \sqrt[3]{x_n + 10}$. Show students that using an initial value for x_1 of either 2 or 3 will lead to a series of numbers that converge eventually to 2.31 . This is the ideal place to discuss simple ideas of convergence and divergence. Students should be aware of the need to write down the result of each iteration, usually to one more place of accuracy than requested for the final answer. Students should be aware that the value of the previous iteration need not be keyed in to the calculator explicitly at each stage; the "ans" key can be used instead.. Students should be given plenty of practice on this procedure. Discuss with the students the technique of setting $x_{n+1} = x_n = L$ (or other convenient letter) in the case of convergence, and using this to deduce the equation for which the result of the iteration is a root. Expressing an equation of the form $x^3=x+10$ as "x=" can be done in many ways, each leading to an iterative formula of the type $x_{n+1} = \mathrm{F}(x_n)$. It is a worthwhile exercise having several of these different iterative formulae available on an OHP and showing students that some of them converge, but that in some cases, the series diverges. The students do not need to know of conditions for convergence. It is also worth demonstrating that a particular iterative formula may give different limiting values for different values of x_0, e.g. the formula $x_{n+1} = \frac{x_n^2 - 1}{2x_n - 4}$ with x_0=0.25 and with x_0=3.75.	OHP showing different iterative formulae for the solution of $x^3=x+10$.	