UNIT 3   Trigonometry and Vectors (P1)

Recommended Prior Knowledge. Students will need an understanding and proficiency in the algebraic techniques from either O' Level Mathematics or IGCSE Mathematics.

Context. This Unit is independent of the other Pure Units, and is ideally left until the end of the course. The work on vectors will however give students taking the Mechanics option at AS a better insight into the work on forces in Unit 4 (Motion and Force) and for schools using two teachers for this syllabus, the work on vectors could be taken from this Unit and used as a preliminary to the mechanics in Units 4 and 5 (Energy, Work and Power).

Outline. The Unit begins with an introduction to the radian as a unit of angle measurement, particularly in work with arc length and area of a sector. The Unit then looks at the three basic trigonometric functions and the graphs, identities and equations linked with them. It concludes with an introduction to vectors in two and three dimensions, including the use of the scalar product.

Topic	Learning Outcomes	Suggested Teaching Activities	Resources	On-Line Resources
4	**Circular Measure** • To understand the definition of a radian, and to use the relationship between radians and degrees. • To use the formulae $s = r\theta$ in solving problems concerning arc length. • To use the formula $A = \dfrac{1}{2}r^2\theta$ in solving problems concerning the sector	Introduce this topic by looking at the function keys (DRG) on their calculators. Define the radian as the angle subtended by an arc of equal length to the radius and stress the importance of this measurement, particularly in calculus in later work. Show that 1 radian is just less than 60°. Discuss and develop the formula $s = r\theta$ for arc length and use this to obtain the relationship that 360° is equivalent to 2π radians. Students should be encouraged to learn the conversion to degrees of the fractional parts of π i.e. $\tfrac{1}{6}\pi, \tfrac{1}{4}\pi, \tfrac{1}{3}\pi$ etc. Discuss and develop the formula $A = \dfrac{1}{2}r^2\theta$ for the area of a sector. If students have used degrees in previous work for	OHP showing a sector of angle 1 radian. Show why this angle is just less than 60°.	www.bbc.co.uk/education/asguru/maths/ → "Pure" → Trigonometry →Radian Measure →Trigonometric functions →Graphs of trigonometric functions →Identities →Trigonometric equations

	• of a circle.	evaluating arc length and sector area, this is an appropriate time to show them how much simpler it is to use radian measure. Exercises on simple sectors should lead to more complicated problems such as finding the perimeter and area of regions such as • a segment • the shape remaining when one sector is removed from a larger sector, both sectors having the same centre and angle • the shape remaining when a sector *AOB* has the triangle *AOC* removed where *C* is the foot of the perpendicular from *A* to *OB* Teachers should be able to find many more situations from either a text-book or from recent P1 examination papers.		
5	**Trigonometry** • To sketch and use graphs of the sine, cosine and tangent functions (for angles of any size, and using either degrees or radians).	All students should be familiar with the basic trigonometry of right -angled triangles and with Pythagoras's Theorem. Most will have met the sine and cosine rules, though this is not needed for this particular topic. Students should have drawn the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ for angles from 0° to 360°, and it should prove a good starting point to revise this work, both with degrees and with radians. Students should be aware of the reflective, rotational and translational properties of the curves, though only the use of these properties is needed.	OHP slides showing the graphs of the sine, cosine and tangent functions for angles from 0^0 to 360^0.	
	• To use the exact values of the sine, cosine and tangent of 30^0, 45^0 and 60^0, and related angles, e.g. $\cos 150^0 = -\tfrac{1}{2}\sqrt{3}$.	Students need to learn the exact values of the sine, cosine and tangent of 30°, 45°and 60° and to be able to calculate, from their knowledge of the basic graphs, the sine, cosine and tangent of related angles such as 120°, 210°, 225° etc.	OHP slides showing the (30°,60°,90°)and the (45°,45°,90°) triangles.	
	• Use the notations $\sin^{-1}x$, $\cos^{-1}x$	Students should be aware of the notation $\sin^{-1}x$, $\cos^{-1}x$ and		

	and tan^{-1}x, to denote the principal values of the inverse trigonometric relations.	tan^{-1}x as a means of expressing an angle. They should be aware that in higher mathematics, particularly in mechanics, angles are often given as, say, sin^{-1}($\frac{2}{5}$) or are required in this form.	
	• Use the identities $\frac{\sin\theta}{\cos\vartheta} \equiv \tan\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$.	Show students the simple proof of $\frac{\sin\theta}{\cos\vartheta} \equiv \tan\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$ by using the definitions of the three trigonometric functions in a right angled triangle. It is a valuable exercise letting students check, on a calculator, that these two identities work for any angle. Introduce students to the idea of an identity and discuss the difference between "equation" and "identity". Discuss the idea of a rigorous proof in mathematics; i.e. starting with one expression and changing it by steps into the required expression (e.g. prove the identity $(1 + \sin x)(\frac{1}{\cos x} - \tan x) = \cos x$). Students find this particular exercise very difficult and lots of practice will be needed with most groups.	OHPs with simple identities.
	• To find all the solutions of simple trigonometric equations lying in a specified interval (general forms of solutions are not included).	General discussion on the solution of a trigonometric equation, starting with simple equations of the form sin$x$=k. Students should appreciate that there are an infinite number of solutions to a trigonometric equation, but that a particular question will only require solutions in a given range, say 0^0 to 360^0. It is worthwhile using different ranges with students, say -180^0 to 180^0 as well as 0^0 to 360^0 and using radian measure 0 to 2π. Extend the work to equations of the form sin2x or sin3x = k. Students should appreciate that there are more solutions in a particular range. Extend this work to a variety of equations that can be solved by	

		using one or both of the basic identities $\dfrac{\sin\theta}{\cos\vartheta} \equiv \tan\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$. Encourage students to realise that any equation within this syllabus will reduce to an equation, or equations, in only one trigonometric function. Link this work with the work covered in Unit 1 on recognition of an equation that is a quadratic in some function of x; e.g. $\sin^2 x + 3\sin x = 2$, or $\cos^2 x + 3\sin x = 2$.		
6	**Vectors** • Use standard notations for vectors, i.e. $\begin{pmatrix} x \\ y \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\mathbf{AB}$, $\mathbf{a}$ • Carry out addition and subtraction of vectors and multiplication of a vector by a scalar, and interpret these operations in geometrical terms. • Use unit vectors, displacement vectors and position vectors. • Calculate the magnitude of a vector.	Most students will have met a few simple ideas on vectors. A general introduction should stress the difference between vector and scalar quantities, and for students studying mechanics, the vector quantities of force and acceleration etc can be mentioned. Ordinary graph paper can be used to give students a good working knowledge of vector notation in two dimensions. They should appreciate that a vector **a**, or $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ does not have a starting point, but that the vector **AB** starts at *A* and finishes at *B*. Using isometric graph paper with directions **a** and **b** is a useful way of illustrating such vectors as **a-b**, **a+b**, **b**-2**a**, 3**a**+4**b**. The discussion should proceed naturally to the ideas of unit vector, position vector and displacement vector and to the calculation of the magnitude of a vector. Students should be able to find a vector of a given length in the direction of a given vector. Discussion should then centre on working in three dimensions and all the above work can be repeated. Students should be able to calculate the length of a line segment *AB* and the position vector of the midpoint of *AB* given the position vectors, or the coordinates, of the points *A* and *B*.	Isometric graph paper, if available.	

	• Calculate the scalar product of two vectors.	The idea of a product of vectors is not a natural one and is difficult to explain to students for the first time. Following the definition of the scalar product of two vectors **a.b** as $\lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \cos \vartheta$, students should realise that • **i.i = j.j = k.k** = 1 and that **i.j = j.k = k.i**= 0 • **a.b** can also be expressed as $a_1 b_1 + a_2 b_2 + a_3 b_3$ • **a.b=b.a** Students will need a lot of practice in evaluating the scalar product of two vectors.	
	• Use the scalar product to determine the angle between two directions.	At this stage, a useful starting point for extending this work to the calculation of an angle in three dimensions is to present students with a triangle *ABC* in which the coordinates of *A*, *B* and *C* (or the position vectors of *A*, *B* and *C*) are given. Discussion should then follow as to how the scalar product of the vectors **AB** and **AC** can be used to find the value of angle *BAC*. The students should be encouraged to find angle *B* (from **BA** and **BC)** and to find angle *C* (from **CA** and **CB**). They should check that the total is 180°. They should be encouraged to repeat this exercise with different triangles.	Have ready OHP slides with examples showing geometric figures in 3 dimensions. There have been several such examples on recent examination papers.
	• Use the scalar product to solve problems concerning perpendicularity of vectors.	Discussion can now centre on the significance of **a.b**=0. Students should be able to use this result to test whether a particular triangle is right angled or to solve other similar geometric problems in two or three dimensions. They should be encouraged to find angles in three-dimensional shapes (cubes, cuboids, pyramids, prisms etc) by use of the appropriate scalar product.	