

## UNIT 1 Algebra (P1)

**Recommended Prior Knowledge.** Students will need an understanding and proficiency in the algebraic techniques from either O Level Mathematics or IGCSE Mathematics.

**Context.** The concepts of algebra as encountered in this unit are fundamental to success in mathematics. A good grasp of these techniques is essential if more advanced units are to be fully understood and implemented. This unit must precede other units, though the work on arithmetic and geometric series can be left until later if other demands (say the need for calculus in mechanics) are present.

**Outline.** The unit begins with an in-depth study of the quadratic function, from both an algebraic and graphical point of view. It looks at the different methods of solving both quadratic equations and inequalities and the solution of simultaneous equations where one equation is quadratic (Topic 1). It considers “functions” in detail, looking at the ideas of domain, range, one-one functions, inverse functions and composition of functions (Topic 2). Because of the algebraic nature of the work on series, Topic 7 is included within this unit and includes a study of arithmetic and geometric series and the binomial expansion  $(a+b)^n$  where  $n$  is a positive integer.

Topic	Learning Outcomes	Suggested Teaching Activities	Resources	On-Line Resources
1	<p><b>The Quadratic Function</b></p> <ul style="list-style-type: none"> <li>To carry out the process of completing the square for a quadratic polynomial <math>ax^2+bx+c</math>.</li> <li>Use this form e.g. to locate the vertex of the graph of <math>y= ax^2+bx+c</math>, or to sketch the graph.</li> </ul>	<p>All students will have drawn graphs of quadratics from tables. Encourage students to recognise the symmetry of the curve and to recognise that graphs of the equation <math>y=ax^2+bx+c</math> produce “U-shaped” parabolas for <math>a&gt;0</math> and “inverted” parabolas for <math>a &lt; 0</math>. Students should appreciate that all quadratic curves have a line of symmetry at <math>x=h</math>, where <math>(h, k)</math> is the maximum or minimum point of the curve.</p> <p>General discussion on why <math>x^2+ax+b</math> can always be expressed as <math>(x+h)^2+k</math>. Extend the discussion to expressing <math>ax^2+bx+c</math> in the form <math>a(x+h)^2+k</math>. Give students a lot of practice with different quadratic expressions including the cases where <math>a &lt; 0</math> and <math>a&gt;1</math>. Discuss with students the advantages of using the “completed-square” expression for locating the vertex of the graph. Lots of practice with sketching graphs of quadratics. (n.b. this technique should be revised when stationary points are covered in the</p>	<p>Graph paper.</p> <p>OHP slides linking “completing the square” with the graphs of quadratics.</p>	<p><a href="http://www.bbc.co.uk/education/asguru/maths">www.bbc.co.uk/education/asguru/maths</a></p> <p>→ “Methods” on the asguru site.</p> <p>→ “Quadratic Functions”</p> <p>→“Completi</p>

	<ul style="list-style-type: none"> <li>To solve quadratic equations.</li> <li>To find the discriminant of a quadratic polynomial <math>ax^2+bx+c</math> and to use this discriminant, e.g. to determine the number of real roots of the equation <math>ax^2+bx+c=0</math>.</li> <li>To solve linear and quadratic inequalities in one unknown.</li> <li>To solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic.</li> </ul>	<p>calculus unit).</p> <p>Most students will have met the formula for solving a quadratic equation. They should be taught to use factors, formula and completing the square, becoming familiar with all three methods of solution. This is an appropriate place to show students how the formula can be derived by expressing <math>ax^2+bx+c</math> in the form <math>a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a^2}</math>.</p> <p>Follow this with a discussion on why the process can fail i.e. <math>b^2 - 4ac &lt; 0</math>. Look at how evaluation of the “discriminant” can determine the number of real roots of an equation. Illustrate graphically the six possible cases, <math>a &gt; 0</math> and <math>b^2 &gt; 4ac</math>, <math>a &lt; 0</math> and <math>b^2 &gt; 4ac</math>, <math>a &gt; 0</math> and <math>b^2 = 4ac</math>, <math>a &lt; 0</math> and <math>b^2 = 4ac</math>, <math>a &gt; 0</math> and <math>b^2 &lt; 4ac</math> and <math>a &lt; 0</math> and <math>b^2 &lt; 4ac</math>.</p> <p>General discussion on the difference between “<math>ax+b = c</math>” and “<math>ax+b &gt; c</math>”. Look at how the techniques for solving an inequality differ from those for solving an equation (in particular how an inequality is affected by multiplication or division by a negative quantity). Students need to be aware that <math>a &lt; b</math> implies <math>b &gt; a</math>.</p> <p>Follow this with a discussion on the quadratic inequality, starting with a quadratic such as <math>(x-2)(x-4) &gt; 0</math>. Textbooks look at several different methods but probably the most straightforward is to mark the critical values (2 and 4 in this case) on a number line segment and to consider the sign of the quadratic function <math>((x-2)(x-4))</math> in each of the intervals <math>x &lt; 2</math>, <math>2 &lt; x &lt; 4</math> and <math>x &gt; 4</math> in this case. Students should realise that the quadratic may not factorise and that the coefficient of <math>x^2</math> may not necessarily be 1. They should be able to cope with the implication of “<math>&lt;</math>” and of “<math>\leq</math>”.</p> <p>Following revision of techniques for solving two simultaneous linear equations, extend this to the solution, by substitution, of two simultaneous equations, where one is linear and one is quadratic. Use OHPs to illustrate these</p>	<p>OHP with diagrams showing the intersection of graphs of linear and</p>	<p>ng the Square”</p> <p>→ “Quadratic Solutions”</p> <p>→ “Inequalities”</p>
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	<ul style="list-style-type: none"> <li>To recognise and solve equations in <math>x</math> which are quadratic in some function of <math>x</math> e.g. <math>x^4 - 5x^2 + 4 = 0</math>.</li> </ul>	<p>solutions. Use quadratic equations of the form <math>y = ax^2 + bx + c</math>, <math>x^2 + y^2 = k</math>, <math>xy = k</math> etc.</p> <p>Look at the general case of a line, say <math>y = mx + 2</math>, meeting a curve, say <math>y = x^2</math>. Show, by means of an OHP, the three geometrical cases that can occur for different values of <math>m</math> "i.e. two points of intersection, the line being a tangent to the curve, and the line failing to intersect the curve". Show how this links with the discriminant <math>b^2 - 4ac</math>. This should link up with the work on coordinate geometry in Unit 2 on "relationships between points of intersection of graphs and solutions of equations".</p> <p>Students should appreciate that many equations reduce to a quadratic equation by means of a simple substitution. Such equations as <math>x^4 - 6x^2 + 8 = 0</math> or <math>x - 6\sqrt{x} + 8 = 0</math> or <math>2^{2x} - 6(2^x) + 8 = 0</math> or <math>2\sin^2 x - 3\sin x + 1 = 0</math> should be recognised and solved as quadratics.</p>	<p>quadratic functions.</p> <p>OHP showing the geometrical links with <math>b^2 - 4ac &lt; 0</math> or <math>= 0</math> or <math>&gt; 0</math> for the solution of a linear with a quadratic.</p>	<p>→ "Simultaneous Equations"</p>
2	<p><b>Functions</b></p> <ul style="list-style-type: none"> <li>To understand the terms function, domain, range, one-one function, inverse function.</li> <li>To identify the range of a given function in simple cases.</li> <li>To determine whether a function is one-one, and to find the inverse of a one-one function in simple cases.</li> </ul>	<p>Most students will have met functional notation, but it is worthwhile giving students practice in working with the notation <math>f(x)</math> and solving such equations as <math>f(x) = 0</math> for different functions. The terms domain and range should be introduced. Problems on finding the range for a given domain, especially for non-linear functions, should be discussed and practised.</p> <p>The idea of an inverse function (<math>f^{-1}</math>) should be examined and students should be taught to find the algebraic inverse of functions by any method. The use of completing the square for a quadratic should be used to find the inverse of a quadratic function. This will necessarily introduce the idea of an inverse existing only if the function is one-one. Time should be spent on examining the quadratic function and realising that the existence of an inverse will depend upon the domain. Students should realise, for example, that the</p>	<p>OHP to illustrate functions which do or do not have an inverse.</p> <p>OHP should be available to illustrate that a quadratic only has</p>	<p><a href="http://www.bbc.co.uk/education/asguru/maths">www.bbc.co.uk/education/asguru/maths</a></p> <p>→ Pure.</p> <p>→ <b>Functions (all the ideas in this syllabus are covered in this section)</b></p>

	<ul style="list-style-type: none"> <li>To illustrate in graphical terms the relation between a one-one function and its inverse.</li> <li>To understand the term "composition" of functions and find this for different functions.</li> </ul>	<p>function <math>f(x) = x^2 - 6x + 8</math> has an inverse if, for example, the domain is <math>x \geq 3</math> or <math>x \leq 3</math>. Indeed <math>f(x)</math> has an inverse for any domain consisting of real numbers providing it does not include both <math>3+a</math> and <math>3-a</math> for any non-zero value of <math>a</math>.</p> <p>Students should be encouraged to pick a simple linear function and to find its inverse. They should plot these on the same diagram with the same scales on both axes. General discussion should follow to deduce the geometrical relationship that links the graph of a function with the graph of its inverse.</p> <p>It is worthwhile introducing this topic by using two linear functions, <math>f</math> and <math>g</math>, and asking students to evaluate such expressions as <math>f(g(2))</math> or <math>g(g(2))</math>, <math>g(f^{-1}(2))</math> etc. The difference between <math>fg</math> and <math>gf</math> should be stressed and "<math>fg \neq gf</math>" should be shown by numerical check. Students should also know how to obtain the algebraic form of <math>fg</math>, <math>ff</math> etc and should have practice in solving such equations as <math>fg(x)=k</math>.</p>	<p>an inverse for a restricted domain.</p> <p>OHP showing graphs of functions and their inverses and illustrating the reflective property of these graphs.</p>	
7	<p><b>Series</b></p> <ul style="list-style-type: none"> <li>To use the expansion of <math>(a+b)^n</math>, where <math>n</math> is a positive integer, (knowledge of greatest term and properties of the coefficients are not required, but the notations <math>\binom{n}{r}</math> and <math>n!</math> should be known).</li> </ul>	<p>As an introduction to the binomial expansion, it is worth spending a little time in letting the students obtain the expansions of <math>(a+b)^n</math> for <math>n=1,2,3,4</math> and <math>5</math> for themselves. Discussion should follow as to how the next expansion i.e. for <math>n=6</math> can be obtained. This should lead to a general discussion of Pascal's triangle. If students have never used the factorial function (<math>x!</math>), this should be introduced and students should be shown that the coefficients in the expansion and the terms in Pascal's triangle are equivalent to <math>\binom{n}{r}</math>. (The proof of this is not required).</p> <p>Students should be given the expansion of <math>(a+b)^n</math> for general <math>n</math>. They should be shown how this can be used to find the expansion of such expressions as <math>(2+3x)^5</math> or <math>(1-2x)^6</math> or</p>	<p>OHP showing Pascal's triangle up to row 6, showing the links with <math>\binom{n}{r}</math> and the coefficients in the expansion of <math>(a+b)^n</math>.</p>	

	<ul style="list-style-type: none"> <li>• To recognise an arithmetic progression.</li> <li>• To recognise a geometric progression.</li> </ul> <ul style="list-style-type: none"> <li>• To use the formula for the <math>n</math>th term and the sum of the first <math>n</math> terms to solve problems involving an arithmetic progression.</li> </ul> <ul style="list-style-type: none"> <li>• To use the formula for the <math>n</math>th term and the sum of the first <math>n</math> terms to solve problems involving a geometric progression.</li> </ul>	<p><math>(1+x^2)^4</math> etc. They need practice in such expansions. They should also be able to multiply an expansion by a linear or quadratic function. e.g. finding the term in <math>x^3</math> in the expansion of <math>(1+2x)(1-3x)^5</math>. (Knowledge of the greatest term and properties of the coefficients are not required). Students should understand the meaning of “the coefficient of <math>x^k</math> for a given <math>k</math>”, (including <math>k=0</math> for which they need to understand the meaning of the term “independent of <math>x</math>) and be able to find this coefficient.</p> <p>General discussion on arithmetic and geometric progressions. Students should appreciate the general properties of each progression and be able to recognise both arithmetic and geometric progressions.</p> <p>Students should appreciate that an arithmetic progression can be expressed as “ <math>a, a+d, a+2d</math> etc”. They should appreciate that the <math>n</math>th term is <math>a + (n - 1)d</math> and that the sum of <math>n</math> terms is <math>\frac{n}{2}(a + l)</math> or <math>\frac{n}{2}(2a + (n - 1)d)</math> . (The proof of this last formula is not required). Students should appreciate that the solution of questions on arithmetic progressions usually reduce to forming simultaneous equations for <math>a</math> and <math>d</math>. They will need a lot of practice on different types of questions, including finding the value of <math>n</math> for which the sum is equal to, or greater than, a particular value.</p> <p>Students should appreciate that a geometric progression can be expressed as “ <math>a, ar, ar^2</math> etc “. They should appreciate that the <math>n</math>th term is <math>ar^{n-1}</math> and that the sum of <math>n</math> terms is <math>\frac{a(1-r^n)}{(1-r)}</math> or <math>\frac{a(r^n-1)}{(r-1)}</math> . (The proof of this last</p>		<p><a href="http://www.bbc.co.uk/education/asguru/maths">www.bbc.co.uk/education/asguru/maths</a></p> <p>→ “Pure” → Sequence and Series”</p>
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