General Certificate of Education
January 2008
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 30 January 2008 9.00 am to 10.30 am

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Give a full geometrical description of the transformation represented by each of the following matrices:

(a) $\begin{bmatrix} 0.8 & 0 & -0.6 \\ 0 & 1 & 0 \\ 0.6 & 0 & 0.8 \end{bmatrix}$; *(3 marks)*

(b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. *(2 marks)*

2 It is given that $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ and $\mathbf{c} = \mathbf{i} + 4\mathbf{j} + 28\mathbf{k}$.

(a) Determine:

(i) $\mathbf{a} \cdot \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(iii) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. *(2 marks)*

(b) Describe the geometrical relationship between the vectors:

(i) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$; *(1 mark)*

(ii) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. *(1 mark)*

3 A shear S is represented by the matrix $\mathbf{A} = \begin{bmatrix} p & q \\ -q & r \end{bmatrix}$, where p, q and r are constants.

(a) By considering one of the geometrical properties of a shear, explain why $pr + q^2 = 1$. *(2 marks)*

(b) Given that $p = 4$ and that the image of the point $(-1, 2)$ under S is $(2, -1)$, find:

(i) the value of q and the value of r; *(3 marks)*

(ii) the equation of the line of invariant points of S. *(3 marks)*

4 The matrix $\mathbf{T}$ has eigenvalues 2 and -2, with corresponding eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\3\end{bmatrix}$ respectively.

(a) Given that $\mathbf{T} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix, write down suitable matrices $\mathbf{U}$, $\mathbf{D}$ and $\mathbf{U}^{-1}$. *(3 marks)*

(b) Hence prove that, for all **even** positive integers n,

$$\mathbf{T}^n = \mathrm{f}(n)\ \mathbf{I}$$

where $\mathrm{f}(n)$ is a function of n, and $\mathbf{I}$ is the 2×2 identity matrix. *(5 marks)*

5 A system of equations is given by

$$\begin{aligned} x + 3y + 5z &= -2 \\ 3x - 4y + 2z &= 7 \\ ax + 11y + 13z &= b \end{aligned}$$

where a and b are constants.

(a) Find the unique solution of the system in the case when $a = 3$ and $b = 2$. *(5 marks)*

(b) (i) Determine the value of a for which the system does not have a unique solution. *(3 marks)*

(ii) For this value of a, find the value of b such that the system of equations is consistent. *(4 marks)*

Turn over for the next question

6 (a) The line l has equation $\mathbf{r} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix}$.

(i) Write down a vector equation for l in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$. *(1 mark)*

(ii) Write down cartesian equations for l. *(2 marks)*

(iii) Find the direction cosines of l and explain, geometrically, what these represent. *(3 marks)*

(b) The plane Π has equation $\mathbf{r} = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

(i) Find an equation for Π in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(ii) State the geometrical significance of the value of d in this case. *(1 mark)*

(c) Determine, to the nearest 0.1°, the angle between l and Π. *(4 marks)*

7 The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) (i) Show that

$$\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$$

for some integer k to be determined. *(3 marks)*

(ii) By multiplying the equation in part (a)(i) by $\mathbf{M}^{-1}$, show that

$$\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$$

for constants a and b to be found. *(3 marks)*

(b) (i) Determine the characteristic equation of $\mathbf{M}$ and show that $\mathbf{M}$ has a repeated eigenvalue, 1, and another eigenvalue, 2. *(6 marks)*

(ii) Give a full set of eigenvectors for each of these eigenvalues. *(5 marks)*

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix $\mathbf{M}$. *(3 marks)*

END OF QUESTIONS