General Certificate of Education
June 2008
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 21 May 2008 1.30 pm to 3.00 pm

For this paper you must have:
- a 12-page answer book
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- Show all necessary working; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- The marks for questions are shown in brackets.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.

Answer **all** questions.

1 Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{bmatrix} 7 & 12 \\ 12 & 0 \end{bmatrix}$. *(6 marks)*

2 The vectors **a**, **b** and **c** are given by

$$\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}, \quad \mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{c} = -2\mathbf{i} + t\mathbf{j} + 6\mathbf{k}$$

where t is a scalar constant.

(a) Determine, in terms of t where appropriate:

(i) $\mathbf{a} \times \mathbf{b}$; *(2 marks)*

(ii) $(\mathbf{a} \times \mathbf{b}) \,.\, \mathbf{c}$; *(2 marks)*

(iii) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. *(2 marks)*

(b) Find the value of t for which **a**, **b** and **c** are linearly dependent. *(2 marks)*

(c) Find the value of t for which **c** is parallel to $\mathbf{a} \times \mathbf{b}$. *(2 marks)*

3 The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 4 & 3 & k \end{bmatrix}$, where k is a constant.

Determine, in terms of k where appropriate:

(a) $\det \mathbf{A}$; *(2 marks)*

(b) $\mathbf{A}^{-1}$. *(5 marks)*

4 Two planes have equations

$$\mathbf{r} \cdot \begin{bmatrix} 5 \\ 1 \\ -1 \end{bmatrix} = 12 \quad \text{and} \quad \mathbf{r} \cdot \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = 7$$

(a) Find, to the nearest $0.1°$, the acute angle between the two planes. *(4 marks)*

(b) (i) The point $P(0, a, b)$ lies in both planes. Find the value of a and the value of b. *(3 marks)*

(ii) By using a vector product, or otherwise, find a vector which is parallel to both planes. *(2 marks)*

(iii) Find a vector equation for the line of intersection of the two planes. *(2 marks)*

5 A plane transformation is represented by the 2×2 matrix $\mathbf{M}$. The eigenvalues of $\mathbf{M}$ are 1 and 2, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

(a) State the equations of the invariant lines of the transformation and explain which of these is also a line of invariant points. *(3 marks)*

(b) The diagonalised form of $\mathbf{M}$ is $\mathbf{M} = \mathbf{U}\,\mathbf{D}\,\mathbf{U}^{-1}$, where $\mathbf{D}$ is a diagonal matrix.

(i) Write down a suitable matrix $\mathbf{D}$ and the corresponding matrix $\mathbf{U}$. *(2 marks)*

(ii) Hence determine $\mathbf{M}$. *(4 marks)*

(iii) Show that $\mathbf{M}^n = \begin{bmatrix} 1 & \mathrm{f}(n) - 1 \\ 0 & \mathrm{f}(n) \end{bmatrix}$ for all positive integers n, where $\mathrm{f}(n)$ is a function of n to be determined. *(3 marks)*

Turn over for the next question

6 Three planes have equations

$$\begin{aligned} x + y - 3z &= b \\ 2x + y + 4z &= 3 \\ 5x + 2y + az &= 4 \end{aligned}$$

where a and b are constants.

(a) Find the coordinates of the single point of intersection of these three planes in the case when $a = 16$ and $b = 6$. *(5 marks)*

(b) (i) Find the value of a for which the three planes do not meet at a single point. *(3 marks)*

(ii) For this value of a, determine the value of b for which the three planes share a common line of intersection. *(5 marks)*

7 A transformation T of three-dimensional space is given by the matrix $\mathbf{W} = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}$.

(a) (i) Evaluate $\det \mathbf{W}$, and describe the geometrical significance of the answer in relation to T. *(2 marks)*

(ii) Determine the eigenvalues of $\mathbf{W}$. *(6 marks)*

(b) The plane H has equation $\mathbf{r} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$.

(i) Write down a cartesian equation for H. *(1 mark)*

(ii) The point P has coordinates (a, b, c). Show that, whatever the values of a, b and c, the image of P under T lies in H. *(4 marks)*

8 By considering the determinant

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$$

show that $(x + y + z)$ is a factor of $x^3 + y^3 + z^3 - kxyz$ for some value of the constant k to be determined. *(3 marks)*

END OF QUESTIONS