General Certificate of Education
June 2005
Advanced Level Examination

MATHEMATICS **MFP4**
Unit Further Pure 4

Wednesday 22 June 2005 Afternoon Session

In addition to this paper you will require:
- an 8-page answer book;
- the **blue** AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 30 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MFP4.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.

Information
- The maximum mark for this paper is 75.
- Mark allocations are shown in brackets.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 Solve the simultaneous equations

$$\begin{aligned} 2x + 7y - 3z &= 5 \\ 3x + \ y + 3z &= 10 \\ 8x + 6y \quad &= 7 \end{aligned}$$ *(4 marks)*

2 (a) Find cartesian equations for the line with vector equation

$$\mathbf{r} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 6 \\ 3 \end{bmatrix}$$

giving your answer in the form $\dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2} = \dfrac{z - a_3}{b_3}$. *(3 marks)*

(b) Determine the direction cosines of this line. *(2 marks)*

3 (a) Evaluate the determinant of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{bmatrix}$. *(2 marks)*

(b) A three-dimensional shape S, with volume $12\,\text{cm}^3$, is transformed by a transformation having matrix $\mathbf{X}$. Find the volume of the image of S in the case when:

(i) $\mathbf{X} = \mathbf{M}$; *(2 marks)*

(ii) $\mathbf{X} = \mathbf{M}\,\mathbf{N}^2$, where $\mathbf{N}$ is a 3×3 matrix and $\det \mathbf{N} = \frac{1}{3}$. *(2 marks)*

4 (a) Describe the transformation given by each of the matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ *(4 marks)*

(b) (i) Determine the matrix $\mathbf{AB}$. *(2 marks)*

(ii) Given that $\mathbf{AB}$ represents a rotation, find the axis of rotation and the magnitude of the angle of rotation. *(2 marks)*

5 The points A, B and C have position vectors

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 4\mathbf{k}, \quad \mathbf{b} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{c} = 5\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$$

respectively.

(a) Write down the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. *(2 marks)*

(b) Find, in the form $\mathbf{r} \cdot \mathbf{n} = d$, a vector equation for the plane ABC. *(4 marks)*

(c) Determine, to the nearest 0.1°, the acute angle between the plane ABC and the line with equation $\mathbf{r} = \mathbf{a} + t(5\mathbf{i} + \mathbf{j} + \mathbf{k})$. *(4 marks)*

6 (a) (i) Explain why $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = 0$ for all vectors $\mathbf{a}$ and $\mathbf{b}$. *(2 marks)*

(ii) Hence show that

$$\mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} + \mathbf{a})) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

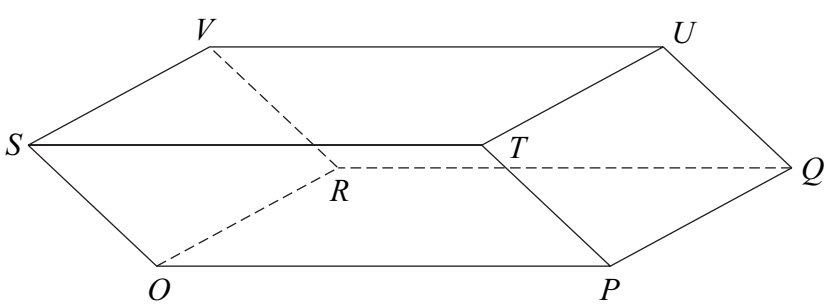
for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. *(2 marks)*

(b) The points P, R and S have position vectors $\mathbf{p} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\mathbf{r} = 2\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$ and $\mathbf{s} = 7\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ respectively, relative to the origin O.

(i) Evaluate $\mathbf{p} \cdot (\mathbf{r} \times \mathbf{s})$. *(2 marks)*

(ii) Explain why $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{s}$ are linearly independent. *(1 mark)*

(iii) The parallelepiped $OPQRSTUV$ is shown in the diagram.



Write down the volume of the parallelepiped. *(1 mark)*

(iv) Use the result of part (a)(ii) to show that

$$\mathbf{p} \cdot (\mathbf{r} \times \mathbf{t}) = \mathbf{p} \cdot (\mathbf{r} \times \mathbf{s})$$

where $\mathbf{t}$ is the position vector of T. *(2 marks)*

7 The transformation T is a stretch in a fixed direction, and maps the point (x, y) to the image point (x', y'), where

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Determine the scale factor of the stretch. *(2 marks)*

(b) By considering the invariant points of T, or otherwise, find an equation for the line of invariant points of T, giving your answer in the form $y = mx$. *(5 marks)*

8 (a) Show that $(a + b + c)$ is a factor of

$$\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ b-c & c-a & a-b \end{vmatrix}$$

Express this determinant as the product of $(a + b + c)$ and a quadratic factor. *(5 marks)*

(b) Hence, or otherwise, show that there is a single real value of a for which the system of equations

$$\begin{aligned} ax + 3y + z &= -5 \\ 4x + (1+a)y + (a+3)z &= 9 \\ 2x + (1-a)y + (a-3)z &= 15 \end{aligned}$$

does not have a unique solution, and find this value of a. *(5 marks)*

9 The matrix $\mathbf{M}$ is given by $\begin{bmatrix} 2 & 7 \\ 4 & k \end{bmatrix}$, where k is a constant. It is given that the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $\mathbf{M}$.

(a) Show that $k = 5$ and find the eigenvalue corresponding to the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. *(3 marks)*

(b) Determine the second eigenvalue of $\mathbf{M}$ and find a corresponding eigenvector. *(5 marks)*

(c) Write down matrices $\mathbf{U}$ and $\mathbf{D}$, having integer elements, such that $\mathbf{M}$ can be expressed in the diagonalised form $\mathbf{M} = \mathbf{UDU}^{-1}$. *(2 marks)*

(d) Write down the matrix $\mathbf{U}^{-1}$. *(1 mark)*

(e) The matrix $\mathbf{M}^{2n}$, for positive integers n, is such that

$$\mathbf{M}^{2n} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for integers a, b, c and d. Show that

$$a = p \times 4^n + q \times 81^n$$

where p and q are rational numbers to be determined. *(4 marks)*

END OF QUESTIONS

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