General Certificate of Education
June 2004
Advanced Level Examination

MATHEMATICS AND STATISTICS (SPECIFICATION B) **MBP6**
Unit Pure 6

Tuesday 29 June 2004 Afternoon Session

In addition to this paper you will require:
- a 12-page answer book;
- the AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 45 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MBP6.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.
- The **final** answer to questions requiring the use of tables or calculators should normally be given to three significant figures.

Information
- The maximum mark for this paper is 80.
- Mark allocations are shown in brackets.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 Show that the curve with equation

$$y = 3\tanh x + 4\,\mathrm{sech}\,x$$

has a stationary point when $\sinh x = \frac{3}{4}$. *(5 marks)*

2 The complex number $\omega = 30\sqrt{31} + 194\,\mathrm{i}$.

(a) Express ω in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and θ is to be given to four decimal places in radians between $-\pi$ and π . *(2 marks)*

(b) The complex number z_1 is the complex number with least positive argument such that ${z_1}^8 = \omega$. Determine $|z_1|$ in an exact form, and find $\arg(z_1)$ to four decimal places. *(2 marks)*

(c) On a diagram of the complex plane, mark the position of z_1, and describe the positions of the remaining seven complex roots of the equation $z^8 = \omega$. *(3 marks)*

3 (a) Find the general solution of the second-order differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 8\mathrm{e}^{3x}$$ *(7 marks)*

(b) Given that $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$, solve this differential equation. *(3 marks)*

4 (a) Prove the identity

$$\sin x + \sin 2x + \sin 3x + \sin 4x \equiv 4\cos\frac{x}{2}\cos x\,\sin\frac{5x}{2}$$ *(5 marks)*

(b) Hence find all solutions in the interval $0 \leqslant x \leqslant \pi$ of the equation

$$\sin x + \sin 2x + \sin 3x + \sin 4x = 0$$

giving your answers in terms of π . *(4 marks)*

5 (a) Use mathematical induction to prove that $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta$ for all positive integers n. *(4 marks)*

(b) (i) Express $(-\sqrt{3} + \mathrm{i})^n$ in the form $2^n(\cos n\theta + \mathrm{i}\sin n\theta)$, giving the value of θ in terms of π. *(3 marks)*

(ii) Hence find the least positive integer value of n for which $(-\sqrt{3} + \mathrm{i})^n$ is a positive real number. *(2 marks)*

6 (a) Determine the eigenvalues and corresponding eigenvectors of the matrix

$$\boldsymbol{M} = \begin{bmatrix} 8 & 6 \\ 6 & 17 \end{bmatrix}$$ *(6 marks)*

(b) The tranformation **T** has matrix $\boldsymbol{M}$.

(i) Show that the invariant lines of **T** are perpendicular. *(2 marks)*

(ii) Give a full geometrical interpretation of **T**. *(3 marks)*

7 (a) Differentiate $\sqrt{1 + t^2}$ with respect to t. *(1 mark)*

(b) (i) Given that $I_n = \int_0^1 \frac{t^n}{\sqrt{1 + t^2}}\,\mathrm{d}t$, where $n \geqslant 0$, show that

$$nI_n = \sqrt{2} - (n - 1)I_{n-2}, \qquad n \geqslant 2$$ *(5 marks)*

(ii) Evaluate I_3 in surd form. *(3 marks)*

(c) Use the substitution $t = \tan\frac{1}{2}x$ to evaluate

$$\int_0^{\frac{\pi}{2}} \sec\tfrac{1}{2}x \tan^3\tfrac{1}{2}x \,\mathrm{d}x$$ *(3 marks)*

TURN OVER FOR THE NEXT QUESTION

8 (a) Using the definitions $\sinh\theta = \frac{1}{2}\left(e^{\theta} - e^{-\theta}\right)$ and $\cosh\theta = \frac{1}{2}\left(e^{\theta} + e^{-\theta}\right)$, prove that:

(i) $2\sinh\theta\cosh\theta \equiv \sinh 2\theta$;

(ii) $2\sinh^2\theta \equiv \cosh 2\theta - 1$. *(4 marks)*

(b) Use the substitution $\cosh\theta = 2x + 1$ to show that:

(i) $\int\sqrt{4x^2 + 4x}\,dx = k\int\sinh^2\theta\,d\theta$, stating the value of k; *(4 marks)*

(ii) $\int\sqrt{4x^2 + 4x}\,dx = \frac{1}{4}(2x+1)\sqrt{4x^2 + 4x} - \frac{1}{4}\cosh^{-1}(2x+1) + C$, where C is an arbitrary constant. *(4 marks)*

(c) Find the length of the arc of the curve with equation

$$y = \frac{1}{4}(2x+1)\sqrt{4x^2 + 4x} - \frac{1}{4}\cosh^{-1}(2x+1)$$

between the points where $x = 77$ and $x = 89$. *(5 marks)*

END OF QUESTIONS