General Certificate of Education
January 2005
Advanced Level Examination

MATHEMATICS AND STATISTICS (SPECIFICATION B) **MBP6**
Unit Pure 6

Tuesday 1 February 2005 Morning Session

In addition to this paper you will require:
- a 12-page answer book;
- the AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed: 1 hour 45 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MBP6.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.
- The **final** answer to questions requiring the use of tables or calculators should normally be given to three significant figures.

Information
- The maximum mark for this paper is 80.
- Mark allocations are shown in brackets.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

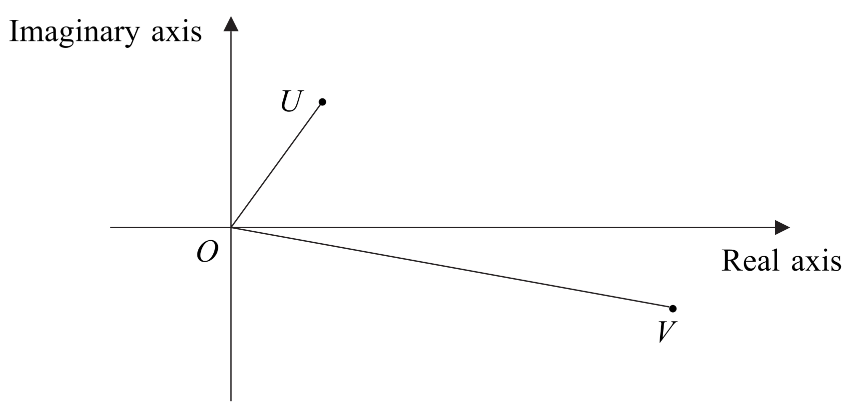
Answer **all** questions.

1 Find the general solution of the differential equation

$$4\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 5y = 0$$ *(5 marks)*

2 Find the area of the region bounded by the curve $y = \cosh x + \operatorname{sech}^2 x$, the x-axis and the lines $x = 0$ and $x = \ln 2$. *(4 marks)*

3 The Argand diagram of the complex plane shows the three points U, V and O which represent the complex numbers u, v and $0 + 0\text{i}$ respectively.



(a) Write down expressions involving u and v which represent:

(i) the length of the line segment UV;

(ii) the angle UOV. *(2 marks)*

(b) On a copy of this Argand diagram, draw the point W which represents the complex number $w = u + v$. *(1 mark)*

4 A curve has parametric equations $x = 2\ln t$, $y = t + \frac{1}{t}$, $t > 0$.

(a) Show that, for this curve, $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{t^2+1}{t^2}\right)^2$. *(4 marks)*

(b) The arc of this curve between the points where $t = 1$ and $t = 2$ is rotated through 2π radians about the x-axis to form a surface of revolution with area S. Find the exact value of S. *(6 marks)*

5 (a) Using the definitions of $\sinh x$ and $\cosh x$ in terms of exponential functions, prove the identity $2\sinh x\cosh x \equiv \sinh 2x$. *(2 marks)*

(b) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\tanh x = \sinh x$$

given that $y = 1$ when $x = 0$. *(7 marks)*

6 (a) Given that $t = \tan\frac{1}{2}x$, show that

$$\sin x\tan\tfrac{1}{2}x + 2\sec x = \frac{2 + 6t^2}{(1 - t^2)(1 + t^2)}$$ *(2 marks)*

(b) Solve the equation

$$2\sin x\tan\tfrac{1}{2}x + 4\sec x + 5 = 0$$

giving all answers for x in radians in the interval $0 < x < 2\pi$. *(6 marks)*

(c) (i) Use the substitution $t = \tan\frac{1}{2}x$ to show that

$$\int_0^{\frac{\pi}{3}} \frac{3}{\sin x\tan\frac{1}{2}x + 2\sec x}\,\mathrm{d}x = \int_0^{\alpha} \frac{3 - 3t^2}{1 + 3t^2}\,\mathrm{d}t$$

stating the exact value of α. *(4 marks)*

(ii) Hence evaluate exactly $\displaystyle\int_0^{\frac{\pi}{3}} \frac{3}{\sin x\tan\frac{1}{2}x + 2\sec x}\,\mathrm{d}x$. *(4 marks)*

7 (a) Determine the real part of $(1 + \mathrm{i}\tan\theta)^3$. *(2 marks)*

(b) Deduce the identity

$$1 - 3\tan^2\theta \equiv \frac{\cos 3\theta}{\cos^3\theta}$$ *(3 marks)*

8 (a) Determine the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{bmatrix} 3 & 5 \\ 2 & 6 \end{bmatrix}$$

(6 marks)

(b) The transformation **T** is given by $x' = 3x + 5y + 1$, $y' = 2x + 6y + 1$.

(i) Show that $(2, -1)$ is a fixed point of **T**. *(2 marks)*

(ii) Hence express **T** in the form

$$\begin{bmatrix} x' - \alpha \\ y' - \beta \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x - \alpha \\ y - \beta \end{bmatrix}$$

for suitable values of the constants α and β. *(1 mark)*

(iii) Hence, using the answer to part (a), find the equation of the line of fixed points of **T**. *(2 marks)*

(c) Show that all lines parallel to $y = x$ are fixed lines of **T**. *(3 marks)*

9 For integers $n \geqslant 0$, let $I_n = \int_0^{\frac{\pi}{3}} e^{3x} \tan^n x \, dx$.

(a) Show that, for $n \geqslant 1$,

$$nI_{n+1} + 3I_n + nI_{n-1} = \left(\sqrt{3}\right)^n e^{\pi}$$

(5 marks)

(b) (i) By considering the cases when $n = 1$ and $n = 3$, or otherwise, show that

$$I_4 + I_3 - 3I_1 = I_0$$

(5 marks)

(ii) Hence evaluate exactly

$$\int_0^{\frac{\pi}{3}} e^{3x} \tan x \left(\tan^3 x + \sec^2 x - 4\right) dx$$

(4 marks)

END OF QUESTIONS