General Certificate of Education
January 2004
Advanced Level Examination

MATHEMATICS (SPECIFICATION A) **MAP4**
Unit Pure 4

Monday 19 January 2004 Morning Session

In addition to this paper you will require:
- an 8-page answer book;
- the AQA booklet of formulae and statistical tables.

You may use a standard scientific calculator **only**.

Time allowed: 1 hour 20 minutes

Instructions
- Use blue or black ink or ball-point pen. Pencil should only be used for drawing.
- Write the information required on the front of your answer book. The *Examining Body* for this paper is AQA. The *Paper Reference* is MAP4.
- Answer **all** questions.
- All necessary working should be shown; otherwise marks for method may be lost.
- The **final** answer to questions requiring the use of tables or calculators should normally be given to three significant figures.
- Tie loosely any additional sheets you have used to the back of your answer book before handing it to the invigilator.

Information
- The maximum mark for this paper is 60.
- Mark allocations are shown in brackets.
- Sheets of graph paper are available on request.

Advice
- Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

Answer **all** questions.

1 (a) Express in the form $a + \mathrm{i}b$:

(i) $(3 + \mathrm{i})^2$; *(1 mark)*

(ii) $(2 + 4\mathrm{i})(3 + \mathrm{i})$. *(1 mark)*

(b) The quadratic equation

$$z^2 - (2 + 4\mathrm{i})z + 8\mathrm{i} - 6 = 0$$

has roots z_1 and z_2.

(i) Verify that $z_1 = 3 + \mathrm{i}$ is a root of the equation. *(2 marks)*

(ii) By considering the coefficients of the quadratic, write down the sum of its roots. *(1 mark)*

(iii) Explain why z_1*, the complex conjugate of z_1, is **not** a root of the quadratic equation. *(1 mark)*

(iv) Find the other root, z_2, in the form $a + \mathrm{i}b$. *(1 mark)*

(c) (i) Label the points representing the complex numbers z_1 and z_2 on an Argand diagram. *(1 mark)*

(ii) Show that $|z_1| = |z_2|$. *(2 marks)*

(iii) Find the value of $\arg\left(\dfrac{z_2}{z_1}\right)$. *(3 marks)*

2 Use de Moivre's Theorem to show that

$$\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right)^7\left(\cos\frac{\pi}{3} - \mathrm{i}\sin\frac{\pi}{3}\right)^5 = -\mathrm{i}.$$ *(6 marks)*

3 The function f is given by

$$f(n) = n^3 + (n+1)^3 + (n+2)^3.$$

(a) Simplify, as far as possible, $f(n+1) - f(n)$. *(4 marks)*

(b) Prove by induction that the sum of the cubes of three consecutive positive integers is divisible by 9. *(5 marks)*

4 (a) Given that

$$y = \sinh^{-1} x,$$

show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}.$$

(3 marks)

(b) The curves C_1 and C_2 have equations $y = \sinh x$ and $y = \sinh^{-1} x$ respectively.

(i) Find the gradient of C_1 and the gradient of C_2 at $x = 0$. *(2 marks)*

(ii) Explain why, for all $x \neq 0$, the gradient of C_1 is greater than 1 and the gradient of C_2 is less than 1. *(3 marks)*

(iii) Sketch on the same axes the graphs of C_1 and C_2. *(2 marks)*

5 A curve C has equation

$$y = \ln(1 - x^2), \qquad 0 \leqslant x < 1.$$

(a) Show that

$$1 + \left(\frac{dy}{dx}\right)^2 = \left(\frac{1+x^2}{1-x^2}\right)^2.$$

(6 marks)

(b) Use the result

$$\frac{1+x^2}{1-x^2} = \frac{2}{1-x^2} - 1$$

to show that the length of the arc of C between the points where $x = 0$ and $x = p$ is

$$2\tanh^{-1} p - p.$$

(4 marks)

6 (a) (i) Verify that $z = 2e^{\frac{1}{4}\pi i}$ is a root of the equation $z^4 = -16$. *(1 mark)*

(ii) Find the other three roots of this equation, giving each root in the form $re^{i\theta}$, where r is real and $-\pi < \theta \leqslant \pi$. *(3 marks)*

(iii) Illustrate the four roots of the equation by points on an Argand diagram. *(2 marks)*

(b) (i) Show that

$$(z - 2e^{\frac{1}{4}\pi i})(z - 2e^{-\frac{1}{4}\pi i}) = z^2 - 2\sqrt{2}\,z + 4.$$ *(3 marks)*

(ii) Express $z^4 + 16$ as the product of two quadratic factors with real coefficients. *(3 marks)*

END OF QUESTIONS